Boundary Layer Approximate Approximations and Cubature of Potentials in Domains

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Abstract

In this article we present a new approach to the computation of volume potentials over bounded domains, which is based on the quasi-interpolation of the density by smooth, almost locally supported basis functions for which the corresponding volume potentials are known. The quasi-interpolant is a linear combination of the basis function with shifted and scaled arguments and with coefficients explicitly given by the point values of the density. Thus, the approach results in semi-analytic cubature formulae for volume potentials, which prove to be high order approximations of the integrals. It is based on multi-resolution schemes for accurate approximations up to the boundary by applying approximate refinement equations of the basis functions and iterative approximations on finer grids. We obtain asymptotic error estimates for the quasi-interpolation and corresponding cubature formulae and provide some numerical examples.

1 Introduction

In recent years the boundary element method (BEM) has been used extensively to solve boundary value problems for partial differential equations with constant coefficients which occur in mechanics, electromagnetics and other fields of mathematical physics.

Let, for example, L be a partial differential operator with known fundamental solution $\mathcal{E}$ and consider the equation

$$Lf = u \quad \text{in } \Omega,$$

complemented with some boundary condition. The simplest way to apply BEM for solving this problem is to represent the solution u as the sum

$$f(\mathbf{x}) = f_0(\mathbf{x}) + Pu(\mathbf{x}),$$

where Pu is the volume potential defined by

$$Pu(\mathbf{x}) = \int_\Omega u(\mathbf{y})\, \mathcal{E}(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$$

and f_0 satisfies the homogeneous equation

$$Lf_0 = 0 \quad \text{in } \Omega,$$

with boundary conditions adjusted such that the total solution f satisfies the boundary condition of the original problem. The remainder f_0 is obtained by solving the corresponding boundary integral equations, involving now the new boundary data for f_0. In order to find these data sufficiently precise, one must be able to compute the volume potential (and, very often, its derivatives) very accurately.

Even more important applications of the volume potentials appear when one combines the BEM with iteration procedures for linear problems with variable coefficients or for non-linear problems. Essentially, the approach for solving boundary problems for nonlinear equations lumps the nonlinearity into body forces and then solves the problem iteratively. This introduces domain integral contributions or volume potentials to the corresponding boundary integral equations.

The construction of closed–form particular solutions is possible only for some special inhomogeneities. Thus the particular solutions must be approximated. However, the direct computation of the potential Pu leads to evaluation of a typically singular integral, which is both numerically expensive and inaccurate if conventional cubature formulae are used.

Therefore, starting with the paper of Nardini/Brebbia [11] it has become increasingly popular to represent the densities u of the volume potentials in terms of simpler functions for which particular solutions are known (see, e.g., [12] and the references therein). Thus, the singularity is removed and one obtains an approximation for the potential Pu. Typically, in the case of volume potentials for isotropic differential operators the most widely used class of approximating functions are special radial basis functions and the approximant interpolates u at certain nodes. Thus, the approximation of the volume potentials turns to the approximation-theoretic problem of the construction of approximants to given functions u by special basis functions and the corresponding error estimates. However, the construction of the interpolant may be rather involved; see for example [13], where the case of Gaussian radial basis functions is studied.

Let us note that another popular method of transforming domain integrals to boundary integrals relies also on the interpolation of the density by linear combinations of certain radial functions (cf. [14] and the references therein).

The aim of this article is to present a new approach to the computation of volume potentials over bounded domains, which is based on the quasi-interpolation of the density u by smooth, almost locally supported basis functions for which particular solutions are known. Since the quasi-interpolant is a linear combination of the basis function with shifted and scaled arguments and with coefficients explicitly given by the point values of u, we get semi-analytic cubature formulae for volume potentials, which prove to be high order approximations of the integrals. Our approach is based on an approximation method proposed by the second author in [2] which use generating functions forming only an *approximate* partition of unity. Given a function u, defined and somewhat regular on $\mathbf{R}^n$, the *approximate approximation* operator $\mathcal{M}_{h,D}$ is defined as the quasi-interpolant

$$\mathcal{M}_{h,D}u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{D}}\right), \qquad (1)$$

where h is the step size, D is a positive parameter and η satisfies some decay and moment conditions. In [7] it is shown that for any integer N it is easy to find a generating function η such that at any point $\mathbf{x}$,

$$|u(\mathbf{x}) - \mathcal{M}_{h,D}u(\mathbf{x})| \le c_{u,\eta}((h\sqrt{D})^N + \varepsilon_0(\eta, D)). \qquad (2)$$

A proper choice of the parameter D allows to make the saturation error $\varepsilon_0(\eta, D)$ as small as necessary, e.g., less than the machine precision.

Formula (1) is the basis of the semi-analytic cubature formulae for the approximation of various integral and pseudo-differential operators. It suffices to find the action of the corresponding operator P on the generating function η of the quasi-interpolant $\mathcal{M}_{h,D}$:

$$Pu(\mathbf{x}) \approx P\mathcal{M}_{h,D}u(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\, P\eta\Big(\frac{\cdot - h\mathbf{m}}{h\sqrt{D}}\Big)(\mathbf{x}).$$

Some important examples are analyzed in [3] and [9], including in particular, the harmonic, elastic, hydrodynamic, diffraction and other potentials.

Such cubature formulae perform well and satisfy estimates similar to (2) only if the approximated function u is defined and somewhat regular on the whole space or can be continued outside the domain of definition with preserved regularity. For functions defined only in bounded domains, we develop multi-resolution schemes for accurate approximation up to the boundary by applying iteratively approximate approximations on finer grids. The mesh refinement is achieved using the analytical factorization of the operator $\mathcal{M}_{h,D}$

$$\mathcal{M}_{h,D} = \mathcal{M}_{\mu h,D}\widetilde{\mathcal{M}}_{h,D}, \quad 0 < \mu < 1,$$

where $\widetilde{\mathcal{M}}_{h,D}$ is another quasi-interpolant of the form (1). These iteration schemes not only retain, but increase the accuracy of approximation at points lying nearer to the boundary. The procedure results in the approximation formula:

$$\mathcal{B}_M u(\mathbf{x}) = \sum_{k=0}^{M} \sum_{\mathbf{m}\in\mathcal{Q}_k} c_{k,\mathbf{m}}\, \eta\left(\frac{\mathbf{x}-h_k\mathbf{m}}{h_k\sqrt{D}}\right), \quad h_k = \mu^k h, \quad 0<\mu<1, \tag{3}$$

which is accurate on the whole of Ω except on a boundary layer of width decreasing exponentially with M, the number of steps made in the iteration scheme from which $\mathcal{B}_M u$ originates. The sets $\mathcal{Q}_k \subset \mathbf{Z}^n$ are such that the mesh points $h_k\mathbf{m} \subset h_k\mathcal{Q}_k$ lie in boundary layers of width exponentially decreasing with k and the coefficients $c_{k,\mathbf{m}}$ are given by

$$c_{k,\mathbf{m}} = \begin{cases} u(h\mathbf{m}) & , \quad k=0, \\ u(h_k\mathbf{m}) - \widetilde{\mathcal{M}}_{h_{k-1},D}u(h_k\mathbf{m}), & k \geq 1. \end{cases}$$

Of course, representation (3) can be used not only near the boundary, but also locally at other regions where higher accuracy is needed.

Clearly, the multi-resolution operator $\mathcal{B}_M$ retains also the quasi-interpolation character of the $\mathcal{M}_{h,D}$ which grants an easy computation of the coefficients $c_{k,\mathbf{m}}$. Moreover, in similarity to wavelet bases and other techniques built upon orthogonal basis functions, the introduction of new higher-frequency terms in (3) does not require re-computation of the coefficients $c_{k,m}$.

The good accuracy provided by (3) for functions on domains can be used to successfully approximate a large class of integral operators. Given an integral operator P with density u defined on a domain, one obtains a cubature formulae for its calculation by setting

$$Pu(\mathbf{x}) \approx P_h u(\mathbf{x} = P\mathcal{B}_M u(\mathbf{x}) = \sum_{k=0}^{M} \sum_{\mathbf{m}\in\mathcal{Q}_k} c_{k,\mathbf{m}}\, P\eta\Big(\frac{\cdot - h_k\mathbf{m}}{h_k\sqrt{D}}\Big)(\mathbf{x}). \tag{4}$$

In the cases of many potentials from mathematical physics, including the harmonic, elastic, hydrodynamic and diffraction potentials, integration can be performed analytically (cf. [2],[3] and [9]). Since the density is reproduced accurately near the boundary if M is large enough, the cubature formula (4) admits error estimates similar to (2). More precisely, in section 7 we prove the following theorem:

Let $u \in W_p^N(\Omega)$ with $N > n/p$ and suppose that P maps $L_p(\mathbf{R}^n)$ into the Bessel potential space $H_p^m(\mathbf{R}^n)$). For any $\varepsilon > 0$ there exists $D > 0$ such that

$$\|Pu - P_h u\|_{H_p^m(\mathbf{R}^n)} \leq c_1 (Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2 h_M^{1/p} \|u\|_{L_\infty(\Omega)} + \varepsilon \|u\|_{W_p^{N-1}(\Omega)}\,.$$

If additionally $P \in \mathcal{L}(H_p^{-m}(\mathbf{R}^n), L_p(\mathbf{R}^n))$ then

$$\|Pu - P_h u\|_{L_p(\mathbf{R}^n)} \leq (c_1 (Dh)^N + c_2 h_M^{1/p+r}) \|u\|_{W_p^N(\Omega)} + \varepsilon\, h^m\, \|u\|_{W_p^{N-1}(\Omega)}\,,$$

where $0 < r < m/n$, $r \leq (p-1)/p$.

We note that a significant reduction of the computational cost can be achieved through anisotropic mesh refinement in direction normal to the boundary which will be studied in a forthcoming paper.

The outline of the paper is as follows. In section 2 we briefly review some results of quasi-interpolation on uniform meshes with smooth and rapidly decaying basis functions. Section 3 is devoted to approximate refinement equations for those functions resulting in the factorization and multiresolution decomposition of the corresponding quasi-interpolation operators. In section 5 we define the boundary layer approximants (3), the approximation errors in integral and weak norms will be studied in section 6. In the final section obtain error estimates for cubature formulae and give examples of semi-analytic cubature for potentials.

2 Approximate approximations on domains

In this section we derive some estimates for the approximation properties of the quasi-interpolant (1) for the case when u is defined on a domain Ω with compact closure and Lipschitz boundary and is continued by zero outside.

2.1 Notation

We will suppose that the generating function η belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ and that for some $N > 0$, the following moment conditions are satisfied:

$$\int_{\mathbf{R}^n} \eta(\mathbf{x})\, d\mathbf{x} = 1, \quad \int_{\mathbf{R}^n} \mathbf{x}^{\boldsymbol{\alpha}} \eta(\mathbf{x})\, d\mathbf{x} = 0, \quad 0 < |\boldsymbol{\alpha}| < N. \tag{5}$$

For a given multi-index $\boldsymbol{\alpha}$, we introduce the numbers

$$\begin{aligned} \varepsilon_{\boldsymbol{\alpha}} = \varepsilon_{\boldsymbol{\alpha}}(\eta, D) &:= D^{-n/2} \Big\| \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big(\frac{\cdot - \mathbf{m}}{\sqrt{D}}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\cdot - \mathbf{m}}{\sqrt{D}}\Big) - \int_{\mathbf{R}^n} \mathbf{x}^{\boldsymbol{\alpha}} \eta(\mathbf{x})\, d\mathbf{x} \Big\|_{L_\infty(\mathbf{R}^n)}, \\ \rho_{\boldsymbol{\alpha}} = \rho_{\boldsymbol{\alpha}}(\eta, D) &:= D^{-n/2} \Big\| \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big| \Big(\frac{\cdot - \mathbf{m}}{\sqrt{D}}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\cdot - \mathbf{m}}{\sqrt{D}}\Big) \Big| \Big\|_{L_\infty(\mathbf{R}^n)}. \end{aligned} \tag{6}$$

From Poisson's summation formula one obtains immediately

$$\varepsilon_{\boldsymbol{\alpha}} \le \sum_{\mathbf{m}\neq\mathbf{0}} |\mathcal{F}_{\mathbf{x}\mapsto\boldsymbol{\xi}}(\mathbf{x}^{\boldsymbol{\alpha}}\eta(\mathbf{x}))(\sqrt{D}\boldsymbol{\nu})|, \quad 0 \le |\boldsymbol{\alpha}| < N, \tag{7}$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}u(\boldsymbol{\xi}) = \int_{\mathbf{R}^n} u(\mathbf{x})\, e^{-2\pi i \langle \mathbf{x}, \boldsymbol{\xi}\rangle}\, d\mathbf{x},$$

We define also the monotone function

$$g_{\boldsymbol{\alpha},D}(t) = D^{-n/2} \sup_{\mathbf{x}\in\mathbf{R}^n} \sum_{|\mathbf{x}-\mathbf{m}|>t} \Big| \Big(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{D}}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{D}}\Big)\Big|,$$

and note that since $\eta \in \mathcal{S}(\mathbf{R}^n)$, $g_{\boldsymbol{\alpha},D}(t)$ decays far out faster than any negative power of t. Of course, if η is continuous, then evidently $\rho_{\boldsymbol{\alpha}}(\eta, D) = g_{\boldsymbol{\alpha},D}(0)$.

For $r > 0$, let $B(\mathbf{x}, r)$ be the closed ball centered at $\mathbf{x}$ of radius r. Finally, if Ω is a bounded domain in $\mathbf{R}^n$, we define the subdomain Ω_r and the equidistant r-neighbourhood Ω_r^+ of Ω by

$$\Omega_r = \{\mathbf{x} : B(\mathbf{x}, r) \subset \Omega\}, \qquad \Omega_r^+ = \{\mathbf{x} : \operatorname{dist}(\mathbf{x}, \Omega) < r\}. \tag{8}$$

2.2 Accuracy of approximate approximation in domains

In [7] it is shown that if u if N-times differentiable and the generating function η satisfies the moment conditions (5), the quasi-interpolant $\mathcal{M}_{h,D}u$ approximates u at a rate $\mathcal{O}(\varepsilon_0 + (h\sqrt{D})^N)$. The quantity ε_0, defined by (6), is referred to as the *saturation error.*

Since $\eta \in \mathcal{S}(\mathbf{R}^n)$, by (7) the values of $\varepsilon_{\boldsymbol{\alpha}}$, $0 \le |\boldsymbol{\alpha}| < N$, can be made as small as needed if D is chosen large enough. Note also that the bound (7) for the saturation error is independent of the step size h.

Clearly, the boundedness of $\Omega = \operatorname{supp} u$ does not imply boundedness of the support of $\mathcal{M}_{h,D}u$. Nevertheless, as η is in the Schwartz class, $\mathcal{M}_{h,D}u(\mathbf{x})$ decays fast with the $\operatorname{dist}(\mathbf{x}, \operatorname{supp} u)$:

Lemma 1 *Suppose that u is a bounded function and $\Omega = \operatorname{supp} u$. Then*

$$|\mathcal{M}_{h,D}u(\mathbf{x})| \leq g_{0,D}(h^{-1}\operatorname{dist}(\mathbf{x},\Omega))\,\|u\|_\infty.$$

Since $g_{0,D} \in \mathcal{S}$, one can find a number $N_s > 0$, such that

$$g_{\boldsymbol{\alpha},D}(N_s) \leq \varepsilon_{\boldsymbol{\alpha}}(\eta, D), \quad 0 \leq |\boldsymbol{\alpha}| < N. \tag{9}$$

In other words, Lemma 1 assures that if N_s is a positive number such that (9) holds, the essential support of $\mathcal{M}_{h,D}u$ is the $N_s h$-neighbourhood $\Omega^+_{N_s h}$ of Ω, in the sense that

$$|\mathcal{M}_{h,D}u(\mathbf{x})| \leq \varepsilon_0\|u\|_\infty \quad \text{whenever} \quad \mathbf{x} \in \mathbf{R}^n \setminus \Omega^+_{N_s h}. \tag{10}$$

Note also that since $|\mathcal{M}_{h,D}u(\mathbf{x})|$ decays far out more rapidly than any power of $\operatorname{dist}(\mathbf{x},\Omega)$, the quasi-interpolant on $\mathbf{R}^n \setminus \Omega^+_{N_s h}$ is of the order of the saturation error ε_0 even in integral norms.

Remark 1 Another consequence of (10) is that the computation of $\mathcal{M}_{h,D}u$ requires to take only the $(2N_s+1)^n$ summands in (1) for which $|\mathbf{x}/h - \mathbf{m}| \leq N_s$, since the error introduced by neglecting the other terms is smaller than the saturation.

In order to show the approximation properties of $\mathcal{M}_{h,D}$ for functions defined on domains and continued by zero outside, we begin by investigation of the behaviour of the quasi-interpolant under truncation of the summation.

Theorem 1 *Suppose that $\eta \in \mathcal{S}(\mathbf{R}^n)$ satisfies the moment conditions (5) and let $N_s > 0$ be such that (9) holds. If u is N-times continuously differentiable in the ball $B(\mathbf{x}, N_s h)$, then*

$$|(I - \mathcal{M}^{(B)}_{h,D})u(\mathbf{x})| \leq 2\sum_{|\boldsymbol{\alpha}|=0}^{N-1} (h\sqrt{D})^{|\boldsymbol{\alpha}|}\,\frac{\varepsilon_{\boldsymbol{\alpha}}(\eta,D)}{\boldsymbol{\alpha}!}|\partial^{\boldsymbol{\alpha}}u(\mathbf{x})|$$
$$+ (h\sqrt{D})^N \sum_{|\boldsymbol{\alpha}|=N} \frac{\rho_{\boldsymbol{\alpha}}(\eta,D)}{\boldsymbol{\alpha}!}\|\partial^{\boldsymbol{\alpha}}u(\mathbf{x})\|_{C(B(\mathbf{x},N_s h))},$$

where $\mathcal{M}^{(B)}_{h,D}$ denotes the truncated quasi-interpolant

$$\mathcal{M}^{(B)}_{h,D} = D^{-n/2}\sum_{h\mathbf{m}\in B(\mathbf{x},N_s h)} u(h\mathbf{m})\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{D}}\Big).$$

PROOF. Set for brevity $B = B(\mathbf{x}, N_s h)$ and $\boldsymbol{\xi}_{\mathbf{m}} = \frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{D}}$. The Taylor expansion of $u(h\mathbf{m})$ around the point $\mathbf{x}$ yields

$$\mathcal{M}^{(B)}_{h,D}u(\mathbf{x}) = D^{-n/2}\sum_{|\boldsymbol{\alpha}|=0}^{N-1}(-\sqrt{D}h)^{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\alpha}}u(\mathbf{x})}{\boldsymbol{\alpha}!}\sum_{h\mathbf{m}\in B}\boldsymbol{\xi}^{\boldsymbol{\alpha}}_{\mathbf{m}}\eta(\boldsymbol{\xi}_{\mathbf{m}})$$
$$+D^{-n/2}\sum_{|\boldsymbol{\alpha}|=N}\frac{(-\sqrt{D}h)^N}{\boldsymbol{\alpha}!}\sum_{h\mathbf{m}\in B}\partial^{\boldsymbol{\alpha}}u(\mathbf{y}_{\mathbf{m}})\,\boldsymbol{\xi}^{\boldsymbol{\alpha}}_{\mathbf{m}}\eta(\boldsymbol{\xi}_{\mathbf{m}}),$$

where $\mathbf{y}_{\mathbf{m}}$ lies on the segment connecting the points $h\mathbf{m}$ and $\mathbf{x}$. If we split the summation over $\mathbf{Z}^n$ and $\mathbf{Z}^n \setminus B$, we obtain for the first inner sum in the right-hand side

$$D^{-n/2}\Big|\sum_{h\mathbf{m}\in B}\boldsymbol{\xi}^{\boldsymbol{\alpha}}_{\mathbf{m}}\eta(\boldsymbol{\xi}_{\mathbf{m}})\Big| \leq \varepsilon_{\boldsymbol{\alpha}}(\eta,D) + g_{\boldsymbol{\alpha},D}(N_s), \quad 0 \leq |\boldsymbol{\alpha}| < N,$$

whereas

$$D^{-n/2}\sum_{h\mathbf{m}\in B}\Big|\boldsymbol{\xi}^{\boldsymbol{\alpha}}_{\mathbf{m}}\eta(\boldsymbol{\xi}_{\mathbf{m}})\Big| \leq \rho_{\boldsymbol{\alpha}}(\eta,D), \quad |\boldsymbol{\alpha}| = N.$$

Choosing N_s as in the statement of the theorem completes the proof. ■

2.3 Examples

As an example, consider the generating functions based on the radial Gaussian

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2}, \quad M = 1, 2, \dots, \tag{11}$$

where $L_k^{(\alpha)}(t)$ denote the generalized Laguerre polynomials defined by

$$L_k^{(\alpha)}(t) = \frac{t^{-\alpha} e^t}{k!} \frac{d^k}{dt^k} \left(t^{k+\alpha} e^{-t} \right), \qquad \alpha > -1. \tag{12}$$

Since the corresponding Fourier transforms are (cf. [3])

$$\mathcal{F}\eta_{2M}(\boldsymbol{\lambda}) = P_{M-1}(\pi^2|\boldsymbol{\lambda}|^2)\, e^{-\pi^2|\boldsymbol{\lambda}|^2}, \quad P_m(t) = \sum_{k=0}^{m} \frac{t^k}{k!}, \tag{13}$$

these functions satisfy the moment conditions (5) with $N = 2M$ and hence, by Theorem 1, give rise to quasi-interpolation formulae (1) of approximate order of convergence $\mathcal{O}((h\sqrt{D})^{2M})$. Furthermore, using (7), the saturation error ε_0 can be estimated by

$$\varepsilon_0(\eta_{2M}, D) \le \sum_{\mathbf{m} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} P_{M-1}(|\mathbf{m}|^2 r^2)\, e^{-|\mathbf{m}|^2 r^2} = \mathcal{O}(r^{2M+n-4}\, e^{-r^2}), \quad r = \pi\sqrt{D}.$$

Note that since $e^{-\pi^2} \approx 5.17 \times 10^{-5}$, already $D = 4$ ensures a saturation error in the range $10^{-15} \div 10^{-12}$ for $1 \le M \le 3$ and space dimensions $n = 2$ and 3.

2.4 L_p-estimates

We recall that our main goal is to use quasi-interpolants for approximation of densities of integral operators, many of which are known to be continuous mappings from L_p to the Sobolev space W_p^l, $l > 0$. Thus, in order to derive estimates for the approximation of the integral operators, it will be necessary to have L_p-estimates for the approximation of the corresponding densities.

By Theorem 1, only the values of the function in a small neighbourhood of the point $\mathbf{x}$ affect the approximation results, and hence, modulo the doubled saturation error, the truncated operator $\mathcal{M}_{h,D}^{(B)}$ possess identical approximation properties as it's untruncated counterpart $\mathcal{M}_{h,D}$. This means also that functions belonging to $C^N(\overline{\Omega})$ are approximated at the rate $\mathcal{O}(\varepsilon_0 + (h\sqrt{D})^N)$ in the subdomain $\Omega_{N_s h}$ (cf. (8)), i.e., at all internal points which lie on a distance larger that $N_s h$ from the boundary $\partial\Omega$. Generally, if u belongs to the Sobolev space $W_p^N(\Omega)$, the following L_p-estimate holds (cf. [9]):

Theorem 2 *Suppose that $\eta \in \mathcal{S}(\mathbf{R}^n)$ satisfies the moment conditions* (5) *and that N_s is as in* (9)*. Further, let Ω be a domain in $\mathbf{R}^n$ with compact closure and Lipschitz boundary and $u \in W_p^N(\Omega)$ with $N > n/p$, $1 \le p \le \infty$. Then,*

$$\|(I - \mathcal{M}_{h,D}^{(B)})u\|_{L_p(\Omega_{N_s h})} \le 2 \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (h\sqrt{D})^{|\boldsymbol{\alpha}|} \frac{\varepsilon_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega_{N_s h})}$$
$$+ (h\sqrt{D})^N \sum_{|\boldsymbol{\alpha}|=N} \frac{\rho_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega)},$$

where $\Omega_{N_s h}$ is the sub-domain defined in (8).

We note that under the requirements in Theorem 2 u is continuous on $\overline{\Omega}$ and thus the quasi-interpolant $\mathcal{M}_{h,D}u$ is well-defined. Clearly, if $u \in \mathring{W}_p^N(\Omega)$, then the result of Theorem 2 can be extended to the whole space $\mathbf{R}^n$ instead of $\Omega_{N_s h}$.

In order to estimate the accuracy of approximation of integral operators, besides the bounds inside the domain given by Theorem 2, one needs estimates for the discrepancy $(I - \mathcal{M}_{h,D})u$ on the whole space.

Theorem 3 *Suppose that the conditions of Theorem 2 hold. Then for any $t > 0$,*

$$\|(I - \mathcal{M}_{h,D})u\|_{L_p(\Omega_{th}^+ \setminus \Omega_{N_s h})} \le c_\Omega h^{1/p}(1 + \rho_0(\eta, D))(N_s + t)^{1/p}\|u\|_{L_\infty(\Omega)}$$
$$\|\mathcal{M}_{h,D}u\|_{L_p(\mathbf{R}^n \setminus \Omega_{th}^+)} \le h^{n/p}\,\|g_{0,D}(|\cdot| + t)\|_{L_p(\mathbf{R}^n)}\,\|u\|_{L_\infty(\Omega)},$$

where c_Ω is a constant depending only on the domain Ω.

The proof is based on the following lemma:

Lemma 2 *Suppose that Ω is a domain in $\mathbf{R}^n$ with compact closure and Lipschitz boundary. For $h > 0$, denote by $\mathcal{X}_{S_h}$ the characteristic function of the boundary layer $\{\mathrm{x} \in \Omega : \mathrm{dist}(\mathrm{x}, \partial\Omega) < h\}$. Then, the following estimates hold:*

$$\|\mathcal{X}_{S_h}u\|_{L_p(\Omega)} \le ch^{(t-1)/pt}\|u\|_{L_{pt}(\Omega)}, \quad 1 \le p, t < \infty\,, \tag{14}$$

$$\|\mathcal{X}_{S_h}u\|_{L_p(\Omega)} \le ch^r\|u\|_{W_p^s(\Omega)}, \quad 1 \le p < \infty,\ 0 < r < s/n,\ r \le 1/p\,, \tag{15}$$

$$\|\mathcal{X}_{S_h}u\|_{(W_p^s(\Omega))'} \le ch^r\|u\|_{L_{p/(p-1)}(\Omega)}, \quad 1 \le p < \infty,\quad 0 < r < s/n,\ r \le 1/p\,, \tag{16}$$

with constants depending only on Ω.

Here $(W_p^s(\Omega))'$ denotes the dual space of $W_p^s(\Omega)$ with respect to the L_2 scalar product.

Proof. The first inequality follows from

$$\int_\Omega |\mathcal{X}_{S_h}u|^p\,d\mathrm{x} \le \Big\{\int_\Omega |u|^{pt}\,d\mathrm{x}\Big\}^{1/t}\Big\{\int_{S_h} d\mathrm{x}\Big\}^{(t-1)/t} = (\mathrm{meas}\,S_h)^{(t-1)/t}\,\|u\|_{L_{pt}(\Omega)}^p.$$

To prove (15), we note first that since $u \in W_p^s(\Omega)$, $s > n/p$, then $u \in C(\overline{\Omega})$. Hence

$$\int_\Omega |\mathcal{X}_{S_h}u|^p\,d\mathrm{x} \le \max_{\mathrm{x}\in S_h}\,|u(\mathrm{x})|^p\,\mathrm{meas}\,S_h \le c\,\mathrm{meas}\,S_h\,\|u\|_{W_p^s(\Omega)}^p\,,$$

so that

$$\|\mathcal{X}_{S_h}u\|_{L_p(\Omega)} \le ch^{1/p}\,\|u\|_{W_p^s(\Omega)}.$$

Since evidently $\|\mathcal{X}_{S_h}u\|_{L_p(\Omega)} \le \|u\|_{L_p(\Omega)}$, we obtain by interpolation

$$\|\mathcal{X}_{S_h}u\|_{L_p(\Omega)} \le ch^{\theta/p}\,\|u\|_{W_p^{s\theta}(\Omega)}, \qquad 0 \le \theta \le 1, \quad s > n/p.$$

Setting $r = \theta/n$ yields (15). Finally, since the operator $\mathcal{X}_{S_h}$ is symmetric, there holds

$$\|\mathcal{X}_{S_h}\|_{L_{p/(p-1)}(\Omega)\mapsto(W_p^s(\Omega))'} = \|\mathcal{X}_{S_h}\|_{W_p^s(\Omega)\mapsto L_p(\Omega)},$$

which proves (16) and the lemma. ∎

PROOF OF THEOREM 3. Let for brevity S denote the boundary strip $S = \Omega_{th}^{+} \setminus \Omega_{N_s h}$. Then by the proof of (14)

$$\|(I - \mathcal{M}_{h,D})u\|_{L_p(S)} \le \|(I - \mathcal{M}_{h,D})u\|_{L_\infty(\Omega)}(\text{meas}\, S)^{1/p} \le (1+\rho_0(\eta, D))(\text{meas}\, S)^{1/p}\|u\|_{L_\infty(\Omega)} .$$

To obtain the second estimate in the formulation of the theorem, we note that

$$\|\mathcal{M}_{h,D}u(\mathbf{x})\|^p_{\mathbf{R}^n \setminus \Omega_{th}^{+}} \le \int\limits_{\mathbf{R}^n \setminus \Omega_{th}^{+}} \left(D^{-n/2} \sum_{h\mathbf{m}\in\Omega} \left| u(h\mathbf{m})\, \eta\big(\tfrac{\mathbf{x}/h - \mathbf{m}}{\sqrt{D}}\big) \right| \right)^p d\mathbf{x}$$

$$\le h^n \, \|u\|^p_{L_\infty(\Omega)} \int\limits_{\text{dist}(\boldsymbol{\xi}, h^{-1}\Omega) > t} \left(D^{-n/2} \sum_{\mathbf{m}\in h^{-1}\Omega} \left| \eta\big(\tfrac{\boldsymbol{\xi} - \mathbf{m}}{\sqrt{D}}\big) \right| \right)^p d\boldsymbol{\xi} \, .$$

By the construction of the set Ω_{th}^{+} we have

$$|(\mathbf{x}/h - \mathbf{m})| \ge t + \inf_{\mathbf{y}\in\Omega_{th}^{+}} |h^{-1}(\mathbf{x} - \mathbf{y})|, \quad \mathbf{x} \in \mathbf{R}^n \setminus \Omega_{th}^{+}, \quad h\mathbf{m} \in \Omega,$$

and hence

$$|\boldsymbol{\xi} - \mathbf{m}| \ge t + \text{dist}(\boldsymbol{\xi}, h^{-1}\Omega_{th}^{+}), \qquad \mathbf{m} \in h^{-1}\Omega, \quad \text{dist}(\boldsymbol{\xi}, h^{-1}\Omega) > t.$$

Lemma 1 provides the estimate

$$D^{-n/2} \sum_{\mathbf{m}\in h^{-1}\Omega} \left| \eta\Big(\frac{\boldsymbol{\xi} - \mathbf{m}}{\sqrt{D}}\Big) \right| \le g_{0,D}(t + \text{dist}(\boldsymbol{\xi}, h^{-1}\Omega_{th}^{+})),$$

and therefore

$$\|\mathcal{M}_{h,D}u(\mathbf{x})\|^p_{\mathbf{R}^n \setminus \Omega_{th}^{+}} \le h^n \|u\|^p_{L_\infty(\Omega)} \int\limits_{|\boldsymbol{\xi}|\ge t} \{g_{0,D}(t + |\boldsymbol{\xi}|)\}^p \, d\mathbf{x}.$$

The proof is completed. ■

Combined, Theorems 2 and 3 give L_p-estimates for the approximation error on the whole of $\mathbf{R}^n$. By Theorem 2, the quasi-interpolant $\mathcal{M}_{h,D}u$ is a good approximation of u at internal points, lying at a distance larger than $N_s h$ from the boundary. The error is then of order $\mathcal{O}(\varepsilon_0 + (h\sqrt{D})^N)$ and can be controlled effectively by a proper choice of the step-size h and the parameter D. The second estimate from Theorem 3 assesses the error accumulated outside of the th-neighbourhood of $\text{supp}\, u$. Since $g_{0,D}$ is in the Schwartz class, $\|g_{0,D}(|\cdot| + t)\|_{L_p(\mathbf{R}^n)} \to 0$ more rapidly then any power of t, so this term can be made of the same order of magnitude as, e.g., the saturation error $\varepsilon(\eta, D)$, by choosing t larger.

Thus, the main contribution to the overall error comes from the boundary strip $\Omega_{th}^{+} \setminus \Omega_{N_s h}$, where, by the first estimate in Theorem 3, the error is of order $\mathcal{O}(h^{1/p})$ if u does not vanish on $\partial\Omega$. Clearly, it will be numerically very expensive to make this term small by choosing h smaller, especially in higher space dimensions. In what follows, we concentrate our efforts to build local mesh refinements near points of where the quasi-interpolant $\mathcal{M}_{h,D}u$ does not approximate with satisfactory accuracy, in particular, near the boundary of the domain.

3 Approximate refinement equations

In this section we concentrate on the construction and properties of the cornerstone of approximate multi-resolution techniques, namely, the refinement equations of the type

$$\eta(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \tilde{\eta}(\mu\boldsymbol{\nu})\, \eta(\mathbf{x}/\mu - \boldsymbol{\nu}) + \text{small remainder term}. \tag{17}$$

3.1 Construction

It was proven in [10], that an approximate refinement equation of type (17) is true for $\eta \in \mathcal{S}(\mathbf{R}^n)$ if the Fourier transform $\mathcal{F}\eta \neq 0$ and that $\tilde{\eta}$ can be determined from

$$\mathcal{F}\tilde{\eta}(\boldsymbol{\xi}) = \frac{\mathcal{F}\eta(\boldsymbol{\xi})}{\mathcal{F}\eta(\mu\boldsymbol{\xi})}. \tag{18}$$

More precisely, the following theorem holds:

Theorem 4 *Suppose that* (18) *holds for some positive* $\mu < 1$ *and that* η, $\tilde{\eta}$ *satisfy*

$$\eta \in \mathcal{S}(\mathbf{R}^n), \quad \tilde{\eta} \in \mathcal{S}(\mathbf{R}^n), \qquad \mathcal{F}\eta > 0.$$

Then

$$\eta\Big(\frac{\mathbf{x}}{\sqrt{D}}\Big) = D^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \tilde{\eta}\Big(\frac{\mu\mathbf{m}}{\sqrt{D}}\Big)\eta\Big(\frac{\mathbf{x}-\mu\mathbf{m}}{\mu\sqrt{D}}\Big) + R_{\eta,\mu,D}(\mathbf{x}), \tag{19}$$

where the remainder $R_{\eta,\mu,D} \in \mathcal{S}(\mathbf{R}^n)$ *is given by*

$$R_{\eta,\mu,D}(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} e^{2\pi i\langle\mathbf{x},\mathbf{m}\rangle/\mu} \int_{\mathbf{R}^n} \mathcal{F}\tilde{\eta}(\boldsymbol{\xi})\mathcal{F}\eta(\mu\boldsymbol{\xi}+\sqrt{D}\mathbf{m}))\, e^{2\pi i\langle\boldsymbol{\xi},\mathbf{x}\rangle/\sqrt{D}}\, d\boldsymbol{\xi}. \tag{20}$$

Moreover, for any $\varepsilon > 0$ *there exists* $D = D(\eta,\mu) > 0$ *such that* $|R_{\eta,\mu,D}(\mathbf{x})| < \varepsilon$.

In the sequel, the function $\tilde{\eta}$ defined by (18) will be referred to as the *adjoint function* corresponding to η.

For example, the generating functions (11) based on the Gaussian satisfy the requirements of Theorem 4, since by (13) they possess positive Fourier transforms. The analytic expression of adjoint functions $\tilde{\eta}_2$, $\tilde{\eta}_4$ and $\tilde{\eta}_6$ in the case of one space dimension are:

$$\begin{aligned} &\tilde{\eta}_2(t) = \frac{e^{-t^2/\alpha}}{\sqrt{\alpha\pi}}, \qquad \tilde{\eta}_4(t) = \frac{1}{\mu^2}\Big[\tilde{\eta}_2(t) - \frac{\alpha}{\mu}\mathcal{W}\big(\frac{\sqrt{\alpha}}{\mu}, \frac{t}{\sqrt{\alpha}}\big)\Big], \\ &\tilde{\eta}_6(t) = \frac{1}{\mu^4}\Big\{\tilde{\eta}_2(t) - 2\frac{\alpha}{\mu}\Re\Big[\frac{1+i\mu^2}{\sqrt{1+i}}\,\mathcal{W}\big(\frac{\sqrt{\alpha(1+i)}}{\mu}, \frac{t}{\sqrt{\alpha}}\big)\Big]\Big\}, \end{aligned} \tag{21}$$

where $\alpha = 1 - \mu^2$,

$$\mathcal{W}(z,t) = \frac{e^{-t^2}}{2}\left\{\mathrm{w}\left(i(z+t)\right) + \mathrm{w}\left(i(z-t)\right)\right\},$$

and $\mathrm{w}(z)$ is the scaled complementary error function

$$\mathrm{w}(z) = e^{-z^2}\,\mathrm{erfc}(-iz) = e^{-z^2}\Big(1 - \frac{2}{\sqrt{\pi}}\int_0^{-iz} e^{-t^2}\,dt\Big).$$

Of course, these formulae allow to obtain analytical representations for the adjoint functions in any space dimension when $\eta(\mathbf{x})$ is a product of one dimensional functions:

$$\eta(\mathbf{x}) = \eta_{2M}(x_1)\ldots\eta_{2M}(x_n).$$

Note that for computations we do not need the analytic expression of the functions $\tilde{\eta}$. In the following section we will show that for our purposes it suffices to precompute the values of $\tilde{\eta}$ just in several points, which can be done with some numerical method for inverse Fourier transform.

3.2 Properties of the adjoint function $\tilde{\eta}$

Suppose that in addition to the requirements of Theorem 4, η is subject also to the moment conditions (5). Since these conditions can be rewritten by Fourier transformation as

$$\mathcal{F}\eta(\mathbf{0}) = 1, \qquad \mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}(\mathbf{x}^{\boldsymbol{\alpha}}\eta(\mathbf{x}))(0) = 0, \quad 0 < |\boldsymbol{\alpha}| < N,$$

relation (18) guarantees that they are satisfied by $\tilde{\eta}$ as well. Then, by Theorem 1, $\tilde{\eta}$ gives rise to a quasi-interpolant $\widetilde{\mathcal{M}}_{h,D}$ featuring the same rate of approximate convergence as $\mathcal{M}_{h,D}$, which is generated by η. Hence, in similarity to (9) one can introduce the positive integer $\widetilde{N}_s = \widetilde{N}_s(D)$, so that

$$\tilde{g}_{\boldsymbol{\alpha},D}(\widetilde{N}_s) \le \tilde{\varepsilon}_{\boldsymbol{\alpha}}, \quad 0 \le |\boldsymbol{\alpha}| < N,$$

where

$$\tilde{g}_{\boldsymbol{\alpha},D}(t) = D^{-n/2} \sup_{\mathbf{x}\in\mathbf{R}^n} \sum_{|\mathbf{x}-\mathbf{m}|>t} \Big| \Big(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{D}}\Big)^{\boldsymbol{\alpha}} \tilde{\eta}\Big(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{D}}\Big)\Big|.$$

and $\tilde{\varepsilon}_{\boldsymbol{\alpha}} = \varepsilon_{\boldsymbol{\alpha}}(\tilde{\eta}, D)$ are defined as in (6). The same estimate as (7) holds also in this case, and consequently, the saturation error $\tilde{\varepsilon}_0 \to 0$ as $D \to \infty$ more rapidly than any power of D. For example, for the adjoint functions $\tilde{\eta}_{2M}$ to η_{2M} (cf. (11)), one obtains by (13) and (18) that

$$\tilde{\varepsilon}_{\mathbf{0}} \le \sum_{\mathbf{m}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} \frac{P_{M-1}(|\mathbf{m}|^2 r^2)}{P_{M-1}(\mu|\mathbf{m}|^2 r^2)}\, e^{-(1-\mu^2)|\mathbf{m}|^2 r^2} = \mathcal{O}\big(r^{2M+n-4}\, e^{-(1-\mu^2)r^2}\big), \quad r = \pi\sqrt{D}.$$

3.3 Quasi-interpolants based on the remainder term

In the following we meet quasi-interpolants generated by the remainder term $R_{\eta,\mu,D}(\mathbf{x})$ of the form

$$\mathcal{R}_{h,D}u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(\mathbf{m}h)\, R_{\eta,\mu,D}(\mathbf{x}/h - \mathbf{m}) \tag{22}$$

By Theorem 4 these quasi-interpolants are properly defined, since we have rapid decay in $\mathbf{x}$. For instance, when η is the Gaussian, the corresponding function $\tilde{\eta}_2$ is by (21) also a scaled Gaussian:

$$\tilde{\eta}_2(\frac{\mathbf{x}}{\sqrt{D}}) = \eta_2(\frac{\mathbf{x}}{\sqrt{D(1-\mu^2)}}) = \eta_2(\frac{\mathbf{x}}{\sqrt{\widetilde{D}}}), \quad \widetilde{D} = D(1-\mu^2).$$

The approximate refinement equation for this case takes the form

$$e^{-|\mathbf{x}|^2/D} = (\pi\widetilde{D})^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} e^{-|\mu\mathbf{m}|^2/\widetilde{D}}\, e^{-|\mathbf{x}/\mu-\mathbf{m}|^2/D} + R_{\eta_2,\mu,D}(\mathbf{x})$$

and the remainder term $R_{\eta_2,\mu,D}(\mathbf{x})$ is given by

$$R_{\eta_2,\mu,D}(\mathbf{x}) = \eta_2\Big(\frac{\mathbf{x}}{\sqrt{D}}\Big)[(I - \widetilde{\mathcal{M}}_{\mu,D})1\,(\mathbf{x}_\mu)] = \eta_2\Big(\frac{\mathbf{x}}{\sqrt{D}}\Big)[(I - \mathcal{M}_{\mu,\widetilde{D}})1\,(\mathbf{x}_\mu)],$$

where $\mathcal{M}_{\mu,\widetilde{D}}1$ is the quasi-interpolant $\mathcal{M}_{\mu,\widetilde{D}}u$ for $u(\mathbf{x}) \equiv 1$ and $\mathbf{x}_\mu = (1-\mu^2)\mathbf{x}$. Thus by Theorem 1 $|R_{\eta_2,\mu,D}(\mathbf{x})| \le \tilde{\varepsilon}_0$ and the quasi-interpolant $\mathcal{R}_{h,D}u$ satisfies the uniform bound

$$|\mathcal{R}_{h,D}u(\mathbf{x})| \le \varepsilon_0(\tilde{\eta}_2, D)\|u\|_{L_\infty} = \varepsilon_0(\eta_2, D(1-\mu^2))\|u\|_{L_\infty}.$$

In following lemma, which we state without proof, we establish the remainder terms in the refinement equations $R_{\eta_{2M},\mu,D}$ for $M > 1$ exhibit similar behaviour as $R_{\eta_2,\mu,D}$:

Lemma 3 *Suppose that η_{2M} is defined by (11) and $0 < \mu < 1$ is a fixed parameter. Then there exist positive univariate polynomials Q_1 and Q_2 of degree $M-1$ such that for any sufficiently large D*

$$|R_{\eta_{2M},\mu,D}(\mathrm{x})| \le Q_1(|\mathrm{x}|^2/D)\, e^{-|\mathrm{x}|^2/D} \sum_{\mathbf{m}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} Q_2(D|\mathbf{m}|^2)\, e^{-\pi^2 D(1-\mu^2)|\mathbf{m}|^2}\,.$$

As a consequence we obtain that the generating function of the quasi-interpolant $\mathcal{R}_{\eta_{2M},\mu,D}$ has amplitude of the same order as the saturation error, and the rate of decay of η_{2M}:

Corollary 1 *Suppose the conditions of Lemma 3 are met. Then, there exists a constant C_R, such that*

$$|R_{\eta_{2M},\mu,D}(\mathrm{x})| \le C_R\, \varepsilon_0(\tilde{\eta}_{2M}, D)\, |\eta(\mathrm{x})|.$$

and, hence, the quasi-interpolant $R_{h,D}u$ defined by (22) admits the uniform estimate

$$|\mathcal{R}_{h,D}u(\mathrm{x})| \le C_R \tilde{\rho}_0 \tilde{\varepsilon}_0.$$

4 Factorization and multiresolution decomposition of quasi-interpolation operators

In this section we use the approximate refinement equation (20) to factorize the quasi-interpolation operator $\mathcal{M}_{h,D}$. Such a factorization allows to obtain an approximate multi-resolution decomposition of the operator on the highest resolution $\mathcal{M}_{\mu^M h,D}$ from which one obtains the desired boundary layer approximate approximation (3) after an appropriate truncation of the summation.

In what follows, we suppose that η and $\tilde{\eta}$ satisfy the requirements of Theorem 4 and the approximate refinement equation (19), and that $\mathcal{M}_{h,D}$, $\tilde{\mathcal{M}}_{h,D}$ are the corresponding quasi-interpolants. Given a sequence of step sizes $\{h_k\}_{k=0}^{M}$, where

$$h_k = \mu^k h, \quad 0 < h, \mu < 1, \quad \mu^{-1} \in \mathbf{Z},$$

we will use the notation

$$\mathcal{A}_k = \mathcal{M}_{\mu^k h,D}, \qquad \tilde{\mathcal{A}}_k = \widetilde{\mathcal{M}}_{\mu^k h,D}, \qquad \mathcal{R}_k = \mathcal{R}_{\mu^k h,D}, \qquad k = 0,1,2\ldots, \tag{23}$$

where $\mathcal{R}_k$ is the quasi-interpolant (22) based on the remainder term in (19).

Theorem 5 (Approximate operator factorization) *Suppose that η and $\tilde{\eta}$ are generating functions satisfying the requirements of Theorem 4 and let $\mathcal{A}_k$, $\tilde{\mathcal{A}}_k$ and $\mathcal{R}_k$ be defined by (23). Then*

$$\mathcal{A}_k = \mathcal{A}_{k+1}\tilde{\mathcal{A}}_k + \mathcal{R}_k, \qquad k = 0,1,2\ldots. \tag{24}$$

Proof. Set for brevity $\eta_D(\mathrm{x}) := D^{-n/2}\eta(\mathrm{x}/\sqrt{D})$ and let $\tilde{\eta}_D$ be the corresponding adjoint function, defined by (18). Then, using the approximate refinement equation (19) one obtains

$$\begin{aligned}\mathcal{A}_k u(\mathrm{x}) &= \sum_{\mathbf{m}\in\mathbf{Z}^n} u(\mathbf{m}h_k)\, \eta_D(\mathrm{x}/h_k - \mathbf{m}) \\ &= \sum_{\boldsymbol{\nu},\mathbf{m}\in\mathbf{Z}^n} u(\mathbf{m}h_k)\, \tilde{\eta}_D(\mu\mathbf{m})\, \eta_D[\mathrm{x}/(\mu h_k) - \mathbf{m}/\mu - \boldsymbol{\nu}] \\ &\qquad + D^{-n/2} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} u(\mathbf{m}h_k)\, R_{\eta,\mu,D}(\mathrm{x}/h_k - \mathrm{x}).\end{aligned}$$

Since μ^{-1} is an integer, $\mathbf{k} = \boldsymbol{\nu} + \mu^{-1}\mathbf{m} \in \mathbf{Z}^n$. Thus, after re-indexing and taking into account that $h_{k+1} = \mu h_k$ one arrives at the representation

$$\mathcal{A}_k u(\mathbf{x}) = \sum_{\mathbf{k},\mathbf{m}\in\mathbf{Z}^n} u(\mathbf{m}h_k)\,\tilde{\eta}_D(\mu\mathbf{k}-\mathbf{m})\,\eta_D[\mathbf{x}/h_{k+1}-\mathbf{k}] + \mathcal{R}_k u(\mathbf{x}).$$

Finally, as $\mu\mathbf{k} = \frac{h_{k+1}\mathbf{k}}{h_k}$, we recognize

$$\mathcal{A}_k u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^n} \tilde{\mathcal{A}}_k u(h_{k+1}\mathbf{k})\,\eta(\mathbf{x}/h_{k+1}-\mathbf{k})$$

which is precisely the claimed identity. ∎

Theorem 6 (Approximate multiresolution decomposition) *Suppose that the approximate operator factorization identity* (24) *holds, and let* $\{\mathcal{X}_k\}_{k=1}^M$ *be a set of linear operators. Then*

$$\mathcal{A}_M\mathcal{X}_M = \mathcal{A}_0\mathcal{X}_0 + \sum_{k=1}^{M}\mathcal{A}_k(\mathcal{X}_k - \tilde{\mathcal{A}}_{k-1}\mathcal{X}_{k-1}) - \sum_{k=0}^{M-1}\mathcal{R}_k\mathcal{X}_k. \tag{25}$$

Proof. By the approximate factorization identity (24) one has

$$\begin{aligned}\mathcal{A}_k\mathcal{X}_k &= \mathcal{A}_{k-1}\mathcal{X}_{k-1} + \mathcal{A}_k\mathcal{X}_k - \mathcal{A}_{k-1}\mathcal{X}_{k-1}\\ &= \mathcal{A}_{k-1}\mathcal{X}_{k-1} + \mathcal{A}_k\mathcal{X}_k - \mathcal{A}_k\tilde{\mathcal{A}}_{k-1}\mathcal{X}_{k-1} - \mathcal{R}_{k-1}\mathcal{X}_{k-1}\\ &= \mathcal{A}_{k-1}\mathcal{X}_{k-1} + \mathcal{A}_k(\mathcal{X}_k - \tilde{\mathcal{A}}_{k-1}\mathcal{X}_{k-1}) - \mathcal{R}_{k-1}\mathcal{X}_{k-1},\end{aligned}$$

and the theorem follows by induction. ∎

Corollary 2 *Under the conditions of Theorem* 6*, suppose that* $\mathcal{X}_k = I$*,* $k = 1,\ldots,M$*. Then*

$$\mathcal{A}_M = \mathcal{A}_0 + \sum_{k=1}^{M}\mathcal{A}_k(I - \tilde{\mathcal{A}}_{k-1}) - \sum_{k=0}^{M-1}\mathcal{R}_k.$$

Adding identity I to both sides in the above corollary and moving $\mathcal{A}_M$ to the right yields

Corollary 3 (Multi-resolution decomposition of identity operator)

$$I = \mathcal{A}_0 + \sum_{k=1}^{M}\mathcal{A}_k(I - \tilde{\mathcal{A}}_{k-1}) + (I - \mathcal{A}_M) - \sum_{k=0}^{M-1}\mathcal{R}_k.$$

5 Boundary layer approximate approximations

In this section we use the multi-resolution decomposition (25) to construct a boundary layer approximate approximation operator $\mathcal{B}_M$. If Ω is a bounded domain and u a sufficiently regular function with $\operatorname{supp} u = \Omega$, then $\mathcal{B}_M u$ is an accurate approximation of u on the whole of Ω except on a thin boundary layer of width decreasing with M. Moreover, the operator $\mathcal{B}_M$ can be defined in such a way that the essential support of $\mathcal{B}_M u$ does not extend outside Ω.

Throughout this section we suppose that $\eta, \tilde{\eta}$ satisfy the requirements of Theorems 1 and 4, and that $\mathcal{M}_{h,D}$ and $\widetilde{\mathcal{M}}_{h,D}$ are the quasi-interpolants generated by η and $\tilde{\eta}$ respectively. Finally, we suppose that there exists a constant C_R, independent of the step size h such that

$$g_{0,R_{\eta,h,D}}(t) = \sup_{\mathbf{x}\in\mathbf{R}^n}\sum_{|\mathbf{x}-\mathbf{m}|>t}|R_{\eta,h,D}(\mathbf{x}-\mathbf{m})| \le C_R\,\tilde{\varepsilon}_0\,g_{0,D}(t), \qquad k = 0,1,2\ldots.$$

For instance, if η is one of the functions defined in (11), such a condition follows from Corollary 1.

We begin by sketching a straightforward way to construct a boundary layer approximate approximation operator $\mathcal{B}_M$ of type (3). Corollary 2 shows that modulo the saturation terms $\sum_{k=0}^{M-1} \mathcal{R}_k$, the multi-resolution operator $\mathcal{A}_0 + \sum_{k=1}^{M} \mathcal{A}_k(I - \mathcal{A}_{k-1})$ performs as the quasi-interpolant $\mathcal{A}_M$ on the finest resolution. Thus, if u is smooth in Ω, the multi-resolution approximation

$$\sum_{k=0}^{M} \mathcal{A}_k \tilde{u}_k = \mathcal{A}_M u + \sum_{k=0}^{M-1} \mathcal{R}_k u, \quad \tilde{u}_k = \begin{cases} u, & k = 0, \\ (I - \tilde{\mathcal{A}}_{k-1})u, & k \geq 1, \end{cases} \tag{26}$$

achieves high accuracy inside and leaves only a thin boundary layer of width $N_s h_M = \mu^M N_s h_0$ where the error is large. Of course, the use of such a scheme is meaningless since one could have applied $\mathcal{A}_M$ at once. Also, its numerical cost of order $\mathcal{O}(h_M^{-n})$ becomes unacceptable if we wish to make the boundary layer very small by making a large number of iterations M. On the other hand, if $u \in C^N(\overline{\Omega})$, Theorem 1 guarantees that

$$|\tilde{u}_k(\mathbf{x})| = \mathcal{O}(\tilde{\varepsilon}_0 + (h_k\sqrt{D})^N)), \quad \mathbf{x} \in \Omega \setminus \Omega_{\widetilde{N}_s h_{k-1}},$$

whereas for points outside the domain, one has

$$|\tilde{u}_k(\mathbf{x})| = |\tilde{\mathcal{A}}_{k-1}u(\mathbf{x})| \leq \tilde{g}_{0,D}(h_{k-1}^{-1} \operatorname{dist}(\mathbf{x}, \Omega)),$$

so $|\tilde{u}_k(\mathbf{x})| \leq \tilde{\varepsilon}_0$ if $\operatorname{dist}(\mathbf{x}, \Omega) > \widetilde{N}_s h_{k-1}$. Hence, if we can truncate those terms in $\mathcal{A}_k \tilde{u}_k$ which contain $\tilde{u}_k(h_k\mathbf{m})$ with argument $h_k\mathbf{m}$ such that $d_{\partial\Omega}(h_k\mathbf{m}) > \tilde{N}_s h_{k-1}$ and neglect the saturation terms, then (26) reduces to the boundary layer approximate approximation (3) with

$$c_{k,\mathbf{m}} = \begin{cases} u(h_0\mathbf{m}), & k = 0, \\ \tilde{u}_k(h_k\mathbf{m}), & k \geq 1, \end{cases}$$

and

$$\mathcal{Q}_k = \begin{cases} \{\mathbf{m} \in \mathbf{Z}^n : \mathbf{m}h_0 \in \Omega\}, & k = 0, \\ \{\mathbf{m} \in \mathbf{Z}^n : d_{\partial\Omega}(\mathbf{x}) \leq \widetilde{N}_s h_{k-1}\}, & k \geq 1. \end{cases}$$

Such a truncation retains the ability of the initial scheme to diminish the remainder boundary layer exponentially with M, while the computational cost is reduced to $\mathcal{O}(h_M^{n-1})$. The price paid is the introduction of an error of order $\mathcal{O}((h_0\sqrt{D})^N))$.

5.1 Boundary layer approximate approximations with support inside Ω

In this section we use Theorem 6 to introduce boundary layer approximate approximations of the type (3) with support essentially contained in the domain of definition Ω of u. Here we use the term "essentially" to describe the fact that $|B_M u(\mathbf{x})|$ is of order $\mathcal{O}(\varepsilon_0 \|u\|_{L_\infty(\overline{\Omega})})$ for $\mathbf{x} \in \partial\Omega$ and decays to zero faster than any negative power of $\operatorname{dist}(\mathbf{x}, \Omega)$ if $\mathbf{x} \in \mathbf{R}^n \setminus \Omega$. Otherwise, if u is smooth enough in Ω, then $\mathcal{B}_M u(\mathbf{x})$ is a high order of (approximate) approximation for $\mathbf{x}$ in $\Omega \setminus S_{M+1}$, where S_{M+1} is a boundary strip of width decreasing exponentially with M.

For $k = 1, 2, \ldots$, we introduce the boundary layers (see Figure 1) in Ω

$$S_k = \begin{cases} \Omega, & k = 0, \\ \Omega \setminus \Omega_{(N_o+N_s)h_{k-1}}, & k \geq 1, \end{cases}$$

where N_o is a free parameter such that

$$N_o > \frac{N_s}{1-\mu}.$$

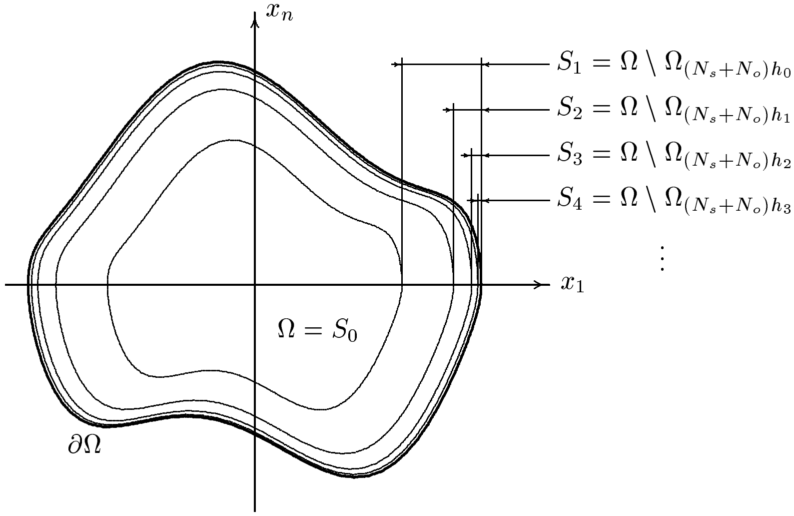

Figure 1: *Illustration for the nested subdomains $\Omega_{(N_s+N_o)h_k}$ and their complements S_k in respect to Ω.*

We define also the operators of multiplication by characteristic functions

$$\mathcal{X}_k^\star u(\mathrm{x}) = \begin{cases} u(\mathrm{x}), & \mathrm{x} \in \overline{\Omega}_{N_o h_k} \\ 0 \ , & \text{otherwise.} \end{cases}$$

and the multi-resolution operator

$$\mathcal{B}_M^\star := \mathcal{A}_0 \mathcal{X}_0^\star + \sum_{k=1}^{M} \mathcal{A}_k (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star),$$

where $\{\mathcal{A}_k\}_0^M$, $\{\tilde{\mathcal{A}}_k\}_0^M$ are the quasi-interpolants from (23). In analogy with the notation in the beginning of 5, one can introduce also discrepancy functions $\tilde{u}_k$, and write

$$\mathcal{B}_M^\star u = \sum_{k=0}^{M} \mathcal{A}_k \tilde{u}_k, \qquad \tilde{u}_k = \begin{cases} \mathcal{X}_0^\star u, & k = 0 \\ (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star) u, & k \geq 1. \end{cases}$$

Now we will show that $\mathcal{B}_M^\star u$ is essentially supported in Ω. We notice first that if $\mathcal{X}$ is a characteristic function of some set, then by Lemma 1, we have

$$|\mathcal{M}_{h,D} \mathcal{X} u(\mathrm{x})| \leq g_{0,D}(h^{-1} \operatorname{dist}(\mathrm{x}, \operatorname{supp} \mathcal{X})).$$

By Theorem 6,

$$\mathcal{B}_M^\star u = \mathcal{A}_M \mathcal{X}_M^\star u + \sum_{k=0}^{M-1} \mathcal{R}_k \mathcal{X}_k^\star u$$

and hence

$$|\mathcal{A}_M \mathcal{X}_M^\star u(\mathrm{x})| \leq g_{0,D}(N_o + h_M^{-1} \operatorname{dist}(\mathrm{x}, \Omega)) \, \|u\|_{C(\overline{\Omega})}, \quad \mathrm{x} \in \mathbf{R}^n \setminus \Omega,$$

as $\operatorname{dist}(\partial\Omega, \operatorname{supp} \mathcal{X}_k) = N_o h_k$ by definition. In other words, $\mathcal{B}_M^\star u|_{\partial\Omega}$ is of the same order as the saturation error if $N_o > N_s$, and $|\mathcal{B}_M^\star u(\mathrm{x})|$ decreases faster than any power of $\operatorname{dist}(\mathrm{x}, \Omega)$ for large x as we declared in the beginning.

In the present form, however, the summation is performed upon the whole of $\mathbf{R}^n$, due to the unbounded support of $\tilde{u}_k$, so it remains to truncate using the idea in the same spirit as we did in the beginning of section 5. In virtue of Theorem 1, $\tilde{u}_k$ is of order $\mathcal{O}(\tilde{\varepsilon} + \sqrt{D} h_{k-1}^N)$ for $\mathrm{x} \in \Omega_{(N_o+\widetilde{N}_s)h_{k-1}}$, so the contribution to $\mathcal{A}_k \tilde{u}_k$ from points in $\Omega_{(N_o+\widetilde{N}_s)h_{k-1}}$ can be neglected. In the following definition, we introduce the operator $\mathcal{B}_M$ in which the summation is performed layer by layer with only minimal overlapping:

Definition 1 *Let $\{\mathcal{X}_k\}_0^M$ be the operator sequence*

$$\mathcal{X}_0 = \mathcal{X}_0^\star, \qquad \mathcal{X}_k u(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega : N_o h_k \le d_{\partial\Omega}(\mathbf{x}) \le (N_o + \widetilde{N}_s) h_{k-1} \\ 0 \ , & \textit{otherwise.} \end{cases}$$

Then the multi-resolution operator

$$\mathcal{B}_M := \mathcal{A}_0 \mathcal{X}_0^\star + \sum_{k=1}^{M} \mathcal{A}_k \mathcal{X}_k (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star). \tag{27}$$

is called the boundary layer approximate approximation operator subordinate to $\{\mathcal{X}_k\}_0^M$.

Alternatively, as we indicated in the beginning of this section, we can rewrite (27) in the form

$$\mathcal{B}_M u(\mathbf{x}) = \sum_{k=0}^{M} \sum_{\mathbf{m} \in \mathcal{Q}_k} c_{k,\mathbf{m}}\, \eta\left(\frac{\mathbf{x} - h_k \mathbf{m}}{h_k \sqrt{D}}\right), \tag{28}$$

with coefficients

$$c_{k,\mathbf{m}} = u_k(h_k \mathbf{m}) = \begin{cases} \mathcal{X}_0^\star u(h_0 \mathbf{m}), & k = 0 \\ (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star) u(h_k \mathbf{m}), & k \ge 1. \end{cases} \tag{29}$$

and

$$\mathcal{Q}_k = \begin{cases} \{\mathbf{m} \in \mathbf{Z}^n : \mathbf{m} h_0 \in \Omega\}, & k = 0, \\ \{\mathbf{m} \in \mathbf{Z}^n : N_o h_k \le d_{\partial\Omega}(\mathbf{x}) \le (N_o + \widetilde{N}_s) h_{k-1}\}, & k \ge 1. \end{cases}$$

Remark 2 The practical implementation of Theorem 6 does not require an explicit formula for $\tilde{\eta}$. Indeed, in order to calculate $\mathcal{B}_M u(\mathbf{x})$ by (28) one has to compute the coefficients $c_{k,\mathbf{m}}$, i.e., to tabulate $(\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star) u$ at the points $h_k \mathbf{m}$ (cf. (29)). By Remark 1, the computation of $A_{k-1}^\star \mathcal{X}_{k-1}^\star u(h_k \mathbf{m})$ requires only summation for indices $\boldsymbol{\nu}$, for which

$$|h_k \mathbf{m}/h_{k-1} - \boldsymbol{\nu}| = |\mu \mathbf{m} - \boldsymbol{\nu}| \le \widetilde{N}_s,$$

where $\widetilde{N}_s$ is such that (9) holds for $\tilde{\eta}$. These $(\mu^{-1}(2N_s + 1))^n$ (or just $\mu^{-1}(2N_s + 1)$, if $\tilde{\eta}$ is a radial function) values can be pre-computed using numerical Fourier inversion of (18).

6 Accuracy

In this section we estimate the error if functions belonging to certain function spaces over Ω are approximated with the operator $\mathcal{B}_M$. Since the cubature formula for the integral operator P is obtained by

$$P u(\mathbf{x}) \approx P \mathcal{B}_M u(\mathbf{x}) = \sum_{k=0}^{M} \sum_{\mathbf{m} \in \mathcal{Q}_k} c_{k,\mathbf{m}}\, P \eta\Big(\frac{\cdot - h_k \mathbf{m}}{h_k \sqrt{D}}\Big)(\mathbf{x}).$$

for the study of the cubature error it is therefore sufficient to estimate $(I - \mathcal{B}_M) u$ in integral norms, for example in L_p or weak Sobolev norms, but on the whole of $\mathbf{R}^n$.

6.1 L_p-estimates

Theorem 7 *Suppose that Ω is a domain in $\mathbf{R}^n$ with compact closure and Lipschitz boundary and let $u \in W_p^N(\Omega)$ with $N > n/p$. For any $\varepsilon > 0$ there exists $D > 0$ and a boundary layer approximation $\mathcal{B}_M$ such that*

$$\|u - \mathcal{B}_M u\|_{L_p(\mathbf{R}^n)} \le c_1 (Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2 (\mu^M h)^{1/p} \|u\|_{L_\infty(\Omega)} + \varepsilon \|u\|_{W_p^{N-1}(\Omega)}\,.$$

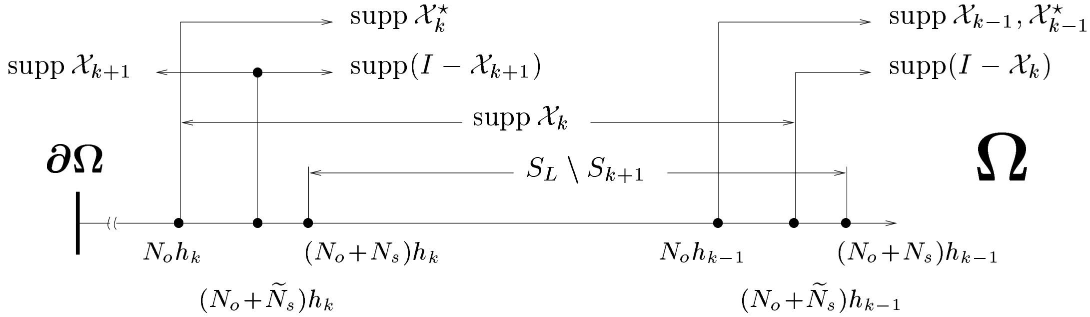

Figure 2: *Sketch of the mutual disposition between the layer $S_k \setminus S_{k+1}$ and the support of the cut-off operators $\mathcal{X}_k$, $\mathcal{X}_k^\star$, $I-\mathcal{X}_k$, etc. The bottom numbers denote distance to the boundary $\partial\Omega$.*

PROOF. We will estimate the L_p norm of $(I-\mathcal{B}_M)u$ on each of the layers $S_L \setminus S_{L+1}$, S_{M+1} (cf. Fig. 2) and on the exterior domain $\mathbf{R}^n \setminus \Omega$.

To estimate $\|\mathcal{B}_M u\|_{L_p(\mathbf{R}^n\setminus\Omega)}$ we decompose $\mathcal{B}_M$ by Theorem 6:

$$\begin{aligned}\mathcal{B}_M &= \mathcal{A}_0\mathcal{X}_0^\star + \sum_{k=1}^{M} \mathcal{A}_k\mathcal{X}_k(\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1}\mathcal{X}_{k-1}^\star)\\ &= \mathcal{A}_0\mathcal{X}_0^\star + \sum_{k=1}^{M}\Big(\mathcal{A}_k\mathcal{X}_k^\star - \mathcal{A}_{k-1}\mathcal{X}_{k-1}^\star + \mathcal{R}_{k-1}\mathcal{X}_{k-1}^\star - \mathcal{A}_k(I-\mathcal{X}_k)(\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1}\mathcal{X}_{k-1}^\star)\Big)\\ &= \mathcal{A}_M\mathcal{X}_M^\star - \sum_{k=1}^{M}\Big(\mathcal{A}_k(I-\mathcal{X}_k)(I-\tilde{\mathcal{A}}_{k-1}) - \mathcal{R}_{k-1}\Big)\mathcal{X}_{k-1}^\star\ ,\end{aligned}$$

where we used in the last equation that $(I-\mathcal{X}_k)\mathcal{X}_k^\star = (I-\mathcal{X}_k)\mathcal{X}_{k-1}^\star$. Thus, by Theorem 3 we get immediately

$$\begin{aligned}&\|\mathcal{B}_M u\|_{L_p(\mathbf{R}^n\setminus\Omega)} \le h_M^{n/p}\|g_{0,D}(|\cdot|+N_o)\|_{L_p(\mathbf{R}^n)}\|u\|_{L_\infty(\Omega)}\\ &+\sum_{k=1}^{M}\Big(h_k^{n/p}\|g_{0,D}(|\cdot|+(N_o+\tilde{N}_s)\mu^{-1})\|_{L_p(\mathbf{R}^n)} + h_{k-1}^{n/p}\|g_{0,R_{\eta,\mu,D}}(|\cdot|+N_o)\|_{L_p(\mathbf{R}^n)}\Big)\|u\|_{L_\infty(\Omega)}\\ &\le c_\mu h^{n/p}\|g_{0,D}(|\cdot|+N_o)\|_{L_p(\mathbf{R}^n)}\|u\|_{L_\infty(\Omega)}\,.\end{aligned} \tag{30}$$

Setting for brevity

$$d_N = N_o + N_s, \qquad \text{and} \qquad \tilde{d}_N = N_o + \tilde{N}_s,$$

we obtain analogously

$$\begin{aligned}&\|u-\mathcal{B}_M u\|_{L_p(S_{M+1})}\\ &\le \|u-\mathcal{A}_M\mathcal{X}_M^\star u\|_{L_p(S_{M+1})} + \sum_{k=1}^{M}\Big(h_k^{n/p}\|g_{0,D}(|\cdot|+\tilde{d}_N\mu^{-1} - d_N\mu^{M-k})\|_{L_p(\mathbf{R}^n)}\\ &\qquad + h_{k-1}^{n/p}\|g_{0,R_{\eta,\mu,D}}(|\cdot|+N_o - d_N\mu^{M+1-k})\|_{L_p(\mathbf{R}^n)}\Big)\|u\|_{L_\infty(\Omega)}\\ &\le \|u-\mathcal{A}_M\mathcal{X}_M^\star u\|_{L_p(S_{M+1})} + h^{n/p}\|u\|_{L_\infty(\Omega)}\sum_{k=1}^{M}\Big(\mu^{kn/p}\|g_{0,D}(|\cdot|+\tilde{d}_N\mu^{-1} - d_N\mu^{M-k}\|_{L_p(\mathbf{R}^n)}\\ &\qquad + \|g_{0,R_{\eta,\mu,D}}(|\cdot|+N_o - d_N\mu^{M+1-k})\|_{L_p(\mathbf{R}^n)})\Big)\,.\end{aligned} \tag{31}$$

To estimate $\|u - \mathcal{B}_M u\|_{L_p(S_L \setminus S_{L+1})}$ we use the representation

$$\mathcal{B}_M = \sum_{k=L+1}^{M} \mathcal{A}_k \mathcal{X}_k (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star) + \mathcal{A}_L \mathcal{X}_L^\star + \sum_{k=1}^{L} \Big(\mathcal{A}_k (I - \mathcal{X}_k)(I - \tilde{\mathcal{A}}_{k-1}) - \mathcal{R}_{k-1} \Big) \mathcal{X}_{k-1}^\star \, .$$

By Theorem 3 we obtain

$$\begin{aligned} &\Big\| \sum_{k=L+1}^{M} \mathcal{A}_k \mathcal{X}_k (\mathcal{X}_k^\star - \tilde{\mathcal{A}}_{k-1} \mathcal{X}_{k-1}^\star) \, u \Big\|_{L_p(S_L \setminus S_{L+1})} \\ &\qquad \le h^{n/p} \|u\|_{L_\infty(\Omega)} \sum_{k=L+1}^{M} \mu^{kn/p} \|g_{0,D}(|\cdot| + d_N \mu^{L-k-1} - \tilde{d}_N \mu^{-1})\|_{L_p(\mathbf{R}^n)} \end{aligned} \tag{32}$$

as well as

$$\begin{aligned} &\| \sum_{k=1}^{L-1} \Big(\mathcal{A}_k (I - \mathcal{X}_k)(I - \tilde{\mathcal{A}}_{k-1}) - \mathcal{R}_{k-1} \Big) \mathcal{X}_{k-1}^\star u \|_{L_p(S_L \setminus S_{L+1})} \\ &\qquad \le h^{n/p} \|u\|_{L_\infty(\Omega)} \sum_{k=1}^{L-1} \Big(\mu^{kn/p} 2 \tilde{\rho}_0 \|g_{0,D}(|\cdot| + N_o - d_N \mu^{k+1-L})\|_{L_p(\mathbf{R}^n)} \\ &\qquad\qquad + \mu^{(k-1)n/p} \|g_{0,R_{\eta,\mu,D}}(|\cdot| + N_o - d_N \mu^{k-L})\|_{L_p(\mathbf{R}^n)} \Big) \, , \end{aligned} \tag{33}$$

showing that these terms are small if N_o and N_s are chosen large enough, and additionally tend to zero together with h.

Consequently, besides the estimate

$$\begin{aligned} \|u - \mathcal{A}_L \mathcal{X}_L^\star u\|_{L_p(S_L \setminus S_{L+1})} &\le (h_L \sqrt{D})^N \sum_{|\boldsymbol{\alpha}|=N} \frac{\rho_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega)} \\ &\quad + 2 \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (h_L \sqrt{D})^{|\boldsymbol{\alpha}|} \frac{\varepsilon_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(S_L \setminus S_{L+1})} \, , \end{aligned}$$

which follows immediately from Theorem 2, it remains to study

$$\|(\mathcal{A}_L (I - \mathcal{X}_L)(I - \tilde{\mathcal{A}}_{L-1}) - \mathcal{R}_{L-1}) \mathcal{X}_{L-1}^\star u\|_{L_p(S_L \setminus S_{L+1})} \, .$$

In view of

$$\begin{aligned} \|\mathcal{R}_{L-1} \mathcal{X}_{L-1}^\star u\|_{L_p(S_L \setminus S_{L+1})} &\le (\operatorname{meas}(S_L \setminus S_{L+1}))^{1/p} \|\mathcal{R}_{L-1} \mathcal{X}_{L-1}^\star u\|_{L_\infty(S_L \setminus S_{L+1})} \\ &\le (\operatorname{meas}(S_L \setminus S_{L+1}))^{1/p} C_R \, \tilde{\varepsilon}_0 \rho_0 \|u\|_{L_\infty(\Omega)} \, , \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\|(\mathcal{A}_L (I - \mathcal{X}_L)(I - \tilde{\mathcal{A}}_{L-1}) - \mathcal{R}_{L-1}) \mathcal{X}_{L-1}^\star u\|_{L_p(\mathbf{R}^n \setminus \Omega_{(N_o + \tilde{N}_s - N_s) h_{L-1}})} \\ &\qquad \le 2 h_L^{n/p} \tilde{\rho}_0 \|g_{0,D}(|\cdot| + N_s)\|_{L_p(\mathbf{R}^n)} \|u\|_{L_\infty(\Omega)} \, , \end{aligned} \tag{35}$$

we are left with the estimation of

$$\begin{aligned} \|\mathcal{A}_L (I - \mathcal{X}_L)(I - \tilde{\mathcal{A}}_{L-1}) \mathcal{X}_{L-1}^\star u\|_{L_p(G_L)} &\le \|\mathcal{A}_L (I - \mathcal{X}_L)(I - \tilde{\mathcal{A}}_{L-1}) \tilde{u}\|_{L_p(G_L)} \\ &\quad + (\operatorname{meas} G_L)^{1/p} \rho_0 \, \|\tilde{g}_{0,D}(|\cdot| + \tilde{N}_s)\|_{L_p(\mathbf{R}^n)} \|u\|_{W_p^N(\Omega)} \, , \end{aligned}$$

where $G_L = S_L \cap \Omega_{(N_o + \tilde{N}_s - N_s) h_{L-1}}$, and $\tilde{u} \in W_p^N(\mathbf{R}^n)$ is the extension of $u \in W_p^N(\Omega)$ with $\|\tilde{u}\|_{W_p^N(\mathbf{R}^n)} = \|u\|_{W_p^N(\Omega)}$.

The function $(I-\mathcal{X}_L)(I-\tilde{\mathcal{A}}_{L-1})\tilde{u}(x)$ is discontinuous on G_L. In order to apply Theorem 2 we introduce the smooth counterpart φ_L of the characteristic function $\mathcal{X}_k$. That means, we require that $\varphi_L \in C_0^N(\mathbf{R}^n)$ is constant with the exception of small neighbourhoods of the jumps of $\mathcal{X}_L$ not containing grid points and that $\varphi_L(h_L\mathbf{m}) = \mathcal{X}_L(h_L\mathbf{m})$, $\mathbf{m}\in\mathbf{Z}^n$. Obviously such a function with

$$\|\partial^{\boldsymbol{\alpha}}\varphi_L\|_{L_\infty} \le c_N h_L^{|\boldsymbol{\alpha}|}\ ,\ 0\le|\boldsymbol{\alpha}|\le N\ ,$$

exists. Furthermore, we introduce the continuous analogue of the quasi-interpolant $\tilde{\mathcal{A}}_{L-1}$

$$\tilde{\mathcal{K}}_{L-1}u(\mathbf{x}) := (\sqrt{D}h_{L-1})^{-n}\int_{\mathbf{R}^n}\tilde{\eta}\Big(\frac{\mathbf{x}-\mathbf{y}}{\sqrt{D}h_{L-1}}\Big)u(\mathbf{y})d\mathbf{y}.$$

and the function $\tilde{U}_L = (I-\varphi_L)(I-\tilde{\mathcal{K}}_{L-1})\tilde{u}$. With this notation we have

$$\mathcal{A}_L(I-\mathcal{X}_L)(I-\tilde{\mathcal{A}}_{L-1})\tilde{u} = \mathcal{A}_L\tilde{U}_L + \mathcal{A}_L(I-\varphi_L)(\tilde{\mathcal{K}}_{L-1}-\tilde{\mathcal{A}}_{L-1})\tilde{u}. \tag{36}$$

and from Theorem 2 we obtain

$$\|\mathcal{A}_L\tilde{U}_L\|_{L_p(G_L)} \le \|\tilde{U}_L\|_{L_p(G_L)} + \|(I-\mathcal{A}_L)\tilde{U}_L\|_{L_p(G_L)} \le \|\tilde{U}_L\|_{L_p(G_L)}$$
$$+ (h_L\sqrt{D})^N\sum_{|\boldsymbol{\alpha}|=N}\frac{\rho_{\boldsymbol{\alpha}}(\eta,D)}{\boldsymbol{\alpha}!}\|\partial^{\boldsymbol{\alpha}}\tilde{U}_L\|_{L_p(\mathbf{R}^n)} + 2\sum_{|\boldsymbol{\alpha}|=0}^{N-1}(h_L\sqrt{D})^{|\boldsymbol{\alpha}|}\,\frac{\varepsilon_{\boldsymbol{\alpha}}(\eta,D)}{\boldsymbol{\alpha}!}\,\|\partial^{\boldsymbol{\alpha}}\tilde{U}_L\|_{L_p(G_L)}\,.$$

Now the rough estimate

$$\|\partial^{\boldsymbol{\alpha}}\tilde{U}_L\|_{L_p(\mathbf{R}^n)} \le C_N\sum_{\boldsymbol{\beta}=0}^{\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!}h_L^{|\boldsymbol{\beta}|-|\boldsymbol{\alpha}|}\|\partial^{\boldsymbol{\beta}}(I-\tilde{\mathcal{K}}_{L-1})\tilde{u}\|_{L_p(\mathbf{R}^n)}$$

together with the moment condition of $\tilde{\eta}$ implies

$$\|\partial^{\boldsymbol{\alpha}}\tilde{U}_L\|_{L_p(\mathbf{R}^n)} \le C_N\|\nabla_N u\|_{L_p(\Omega)}\sum_{\boldsymbol{\beta}=0}^{\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!}h_L^{|\boldsymbol{\beta}|-|\boldsymbol{\alpha}|}(h_{L-1}\sqrt{D})^{N-|\boldsymbol{\beta}|}\int\limits_{\mathbf{R}^n}|\tilde{\eta}(\mathbf{x})||\mathbf{x}|^{|\boldsymbol{\beta}|}\,d\mathbf{x},$$

resulting in

$$\|\partial^{\boldsymbol{\alpha}}\tilde{U}_L\|_{L_p(\mathbf{R}^n)} \le c_{\eta,D}h_{L-1}^N\|\nabla_N u\|_{L_p(\Omega)} \tag{37}$$

with a constant $c_{\eta,D}$ depending only on η, D and μ. The second term in (36) can be written as the difference between an integral operator and its semi-discretization

$$\mathcal{A}_L(I-\varphi_L)(\tilde{\mathcal{K}}_{L-1}-\tilde{\mathcal{A}}_{L-1})\tilde{u} = h_{L-1}^{-n}\int_{\mathbf{R}^n}\Theta_L\Big(\frac{\mathbf{x}}{h_L},\frac{\mathbf{y}}{h_{L-1}}\Big)\tilde{u}(\mathbf{y})d\mathbf{y} - \sum_{\mathbf{j}\in\mathbf{Z}^n}\Theta_L\Big(\frac{\mathbf{x}}{h_L},\mathbf{j}\Big)\tilde{u}(\mathbf{j}h_{L-1})$$

with the smooth kernel function

$$\Theta_L(\mathbf{x},\mathbf{y}) := D^{-n}\sum_{\mathbf{m}\notin h_L^{-1}\,\mathrm{supp}\,\mathcal{X}_L}\eta\Big(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{D}}\Big)\tilde{\eta}\Big(\frac{\mathbf{m}\mu-\mathbf{y}}{\sqrt{D}}\Big)\,.$$

This difference can be estimated by using the Taylor expansion of $\tilde{u}\in W_p^N(\mathbf{R}^n)$ in the following form (cf. [7],[1]):

$$\|\mathcal{A}_L(I-\varphi_L)(\tilde{\mathcal{K}}_{L-1}-\tilde{\mathcal{A}}_{L-1})\tilde{u}\|_{L_p(G_L)} \le c(Dh_{L-1})^N\|\nabla_N u\|_{L_p(\Omega)}$$
$$+\sum_{|\boldsymbol{\alpha}|=0}^{N-1}(Dh_{L-1})^{|\boldsymbol{\alpha}|}\,\|\partial^{\boldsymbol{\alpha}}u\|_{L_p(G_L)}\sum_{\boldsymbol{\beta}=0}^{\boldsymbol{\alpha}}\frac{\rho_{\boldsymbol{\beta}}(\eta,D)\,\varepsilon_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\tilde{\eta},D)}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\,\mu^{\boldsymbol{\beta}}\,.$$

with some constant c not depending on $\tilde{u}$ and h. Summing up the last estimate together with (30)–(35) and (37) we see that for $u \in W_p^N(\Omega)$

$$\|u - \mathcal{B}_M u\|_{L_p(\mathbf{R}^n)} \le \|u - \mathcal{A}_M \mathcal{X}_M^\star u\|_{L_p(S_{M+1})} + c(Dh)^N \|\nabla_N u\|_{L_p(\Omega)}$$
$$+ \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (Dh)^{|\boldsymbol{\alpha}|}\, \delta_{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega)} + h^{n/p} \delta^{(1)} \|u\|_{L_\infty(\Omega)} ,$$

where the numbers $\delta_{\boldsymbol{\alpha}}$, which depend on $\varepsilon_{\boldsymbol{\beta}}$ and $\tilde{\varepsilon}_{\boldsymbol{\beta}}$, can be made arbitrarily small for D large enough, and $\delta^{(1)}$ is determined by the functions g_0 and is sufficiently small if the parameters N_0 and N_s are appropriately chosen. Thus we have only to apply Lemma 2 (see also Theorem 3) and the proof of Theorem 7 is complete. ■

6.2 Pointwise estimates

In a similar way one can show the following pointwise result

Theorem 8 *Suppose that $u \in C^N(\overline{\Omega})$ and the boundary layer approximate approximation operator $\mathcal{B}_L$ is defined by (27). Then for any $\varepsilon > 0$ and $\mathbf{x} \in \Omega \setminus S_{M+1}$, there exist $D > 0$ and positive integers N_s and N_o, such that the accuracy of approximation satisfies the estimate*

$$|(I - \mathcal{B}_M)u(\mathbf{x})| \le c(\sqrt{D} h_k)^N \|\nabla_N u\|_{L_\infty(\Omega)} + \varepsilon \|u\|_{C^{N-1}(\overline{\Omega})},$$

where $0 \le k \le M$ denotes the index for which $\mathbf{x} \in S_k \setminus S_{k+1}$.

Thus the behaviour of $\mathcal{B}_M u(\mathbf{x})$ is actually very close to that of $\mathcal{A}_k u(\mathbf{x})$ for some positive $k \le M$, where k increases as the distance from $\mathbf{x}$ to the boundary decreases. This leads to the effect that the approximation becomes better in points $\mathbf{x} \in \Omega \setminus S_{M+1}$ which lie nearer the boundary $\partial\Omega$.

6.3 Estimates in weak norms

Quasi-interpolation on uniform meshes of the form (1) has the remarkable property that it converges in weak norms, since the saturation error, which is caused by fast oscillating functions, converges weakly to zero. The same property holds for the case of nonuniform meshes considered here. In the proof of Theorem 7, the approximation error $(I - \mathcal{B}_M)u$ was decomposed into

$$(I - \mathcal{B}_M)u = (I - \mathcal{A}_M \mathcal{X}_M^\star)u + \sum_{k=1}^{M} \Big(\mathcal{A}_k (I - \mathcal{X}_k)(I - \tilde{\mathcal{A}}_{k-1}) - \mathcal{R}_{k-1} \Big) \mathcal{X}_{k-1}^\star u \, .$$

The second term consists of functions with L_p-norms which do not exceed $c(Dh)^N \|\nabla_N u\|_{L_p(\Omega)}$ and $h^{n/p} \delta^{(1)} \|u\|_{L_\infty(\Omega)}$, respectively, plus small oscillating functions. Therefore one can show similarly to [7] that for $s > 0$

$$\Big\| \sum_{k=1}^{M} \Big(\mathcal{A}_k (I - \mathcal{X}_k)(I - \tilde{\mathcal{A}}_{k-1}) - \mathcal{R}_{k-1} \Big) \mathcal{X}_{k-1}^\star u \Big\|_{H_p^{-s}} \le c_\eta (Dh)^N \|\nabla_N u\|_{L_p(\Omega)}$$
$$+ h^{n/p} \delta^{(1)} \|u\|_{L_\infty(\Omega)} + c_s\, h^s \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (h\sqrt{D})^{|\boldsymbol{\alpha}|} \frac{\varepsilon_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega)} ,$$

where $H_p^s = H_p^s(\mathbf{R}^n)$ denotes the Bessel potential space equipped with the norm

$$\|u\|_{H_p^s} = \|\mathcal{F}^{-1}(1 + 4\pi^2 |\cdot|)^{s/2} \mathcal{F} u\|_{L_p} = \|(I - \Delta)^{s/2} u\|_{L_p} \, .$$

Thus it remains to estimate $\|(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{H_p^{-s}}$. For integer $s > 0$ we have

$$\|(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{H_p^{-s}} \le c(\|\mathcal{A}_M \mathcal{X}_M^\star)u\|_{L_p(\mathbf{R}^n \setminus \Omega)} + \|(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{(W_q^s(\Omega)')}$$

with $q = p/(p-1)$, and from Lemma 2 one gets for $0 < r < s/n$, $r \le 1/q$

$$\|\mathcal{X}_{S_{M+1}}(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{(W_q^s(\Omega)'} \le c h_M^r \|\mathcal{X}_{S_{M+1}}(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{L_p(\Omega)} \le c h_M^{r+1/p} \|u\|_{W_p^N(\Omega)} .$$

Furthermore,

$$\|(I - \mathcal{X}_{S_{M+1}})(I - \mathcal{A}_M \mathcal{X}_M^\star)u\|_{(W_q^s(\Omega)'} = \sup_{\|\varphi\|_{W_q^s(\Omega)}=1} \Big| \int_{\Omega \setminus S_{M+1}} (I - \mathcal{A}_M \mathcal{X}_M^\star)u \, \varphi \, d\mathbf{x} \Big|$$

$$\le c_\eta (Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_s \, h_M^s \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (h_M \sqrt{D})^{|\boldsymbol{\alpha}|} \frac{\varepsilon_{\boldsymbol{\alpha}}(\eta, D)}{\boldsymbol{\alpha}!} \|\partial^{\boldsymbol{\alpha}} u\|_{L_p(\Omega)} ,$$

so that the following approximation result is valid.

Theorem 9 *Suppose that Ω is a domain in $\mathbf{R}^n$ with compact closure and Lipschitz boundary and let $u \in W_p^N(\Omega)$ with $N > n/p$. Then for any $\varepsilon > 0$ there exists $D > 0$ and a boundary layer approximation $\mathcal{B}_M$ such that*

$$\|u - \mathcal{B}_M u\|_{H_p^{-s}(\mathbf{R}^n)} \le (c_1 (Dh)^N + c_2 (\mu^M h)^{1/p+r}) \|u\|_{W_p^N(\Omega)} + \varepsilon \, h^s \, \|u\|_{W_p^{N-1}(\Omega)} ,$$

where $0 < r < s/n$ and $r \le (p-1)/p$.

6.4 Numerical examples

Here we give some numerical examples to illustrate the overall approximation properties of the operator $\mathcal{B}_M$ defined by (27), and especially the behaviour of the error near the boundary. We shall use the boundary layer approximate approximation (28) generated by the functions η_2, η_4, η_6 based on the Gaussian (see (11)), providing second, fourth, and sixth order of approximate convergence. The corresponding adjoint functions $\tilde{\eta}_2$, $\tilde{\eta}_4$, $\tilde{\eta}_6$ are given by (21). In all cases we use $D = 3$, which assures saturation levels of magnitude 1×10^{-12}, 1×10^{-11} and 1×10^{-10} for quasi-interpolants $\mathcal{M}_{h,D}$ based on η_2, η_4, η_6, respectively. The step refinement ratio in all examples is $\mu^{-1} = 3$.

We recall that by Theorem 8, $\mathcal{B}_M$ performs approximately as $\mathcal{A}_k$ on the k-th boundary strip $S_k \setminus S_{k+1}$, i.e., *the nearer the boundary, the better approximation.* The approximation results are plotted over the boundary layer

$$S_{M+1} \setminus S_0 = \{\mathbf{x} \in \Omega : (N_o + N_s) h_{M+1} \le \operatorname{dist}(\mathbf{x}, \partial\Omega) \le (N_o + N_s) h_0\}$$

in order to illustrate the interplay between the different quasi-interpolants building the operator $\mathcal{B}_M$. Since the step-size used by $\mathcal{B}_M$ is proportional to the distance from the boundary, one can determine the order of the formula used by the slope of the error plot $|(I - \mathcal{B}_M)u|$ against the distance to the boundary in logarithmic scales.

Consider the plot in Fig. 3a showing the error from the approximation of $\cos(1000t)$ near the boundary using the second-order formula based on the Gaussian. One can clearly see the step-wise increase of the accuracy towards the boundary until a saturation is reached. The error remains unchanged within $S_k \setminus S_{k+1}$ for fixed k, since the step does not change there. Observe also the slope of the "staircase" — it is approximately two. In Fig. 3b the same function is approximated using the sixth-order formula based on η_6. Here the slope is approximately $6 : 1$, but the saturation error is higher.

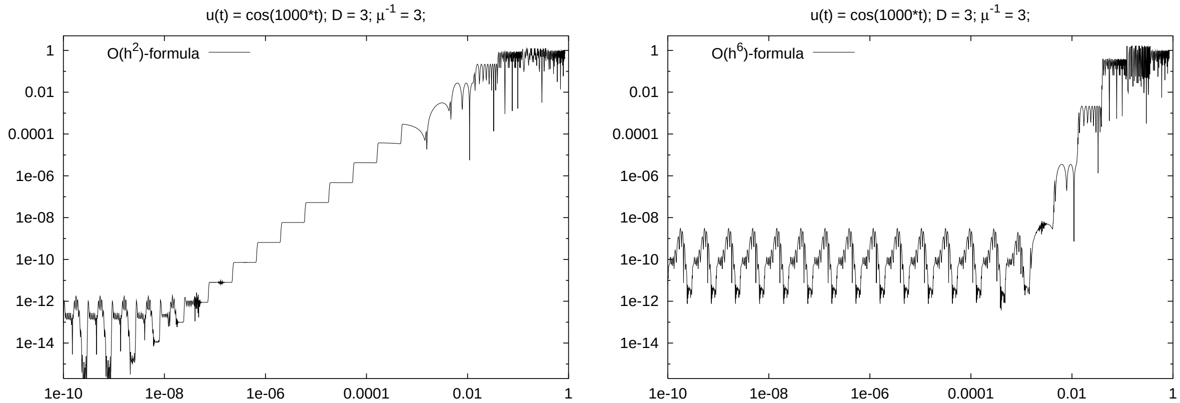

Figure 3: *Boundary layer error plots for* $(I - \mathcal{B}_M)cos(1000t)$ *using* **a)** $\mathcal{O}(h^2)$*-order formula, and* **b)** $\mathcal{O}(h^6)$*-order formula.*

The last example represent boundary error plots for approximation of the function

$$u(x_1, x_2) = \begin{cases} \cos(100\,|\mathbf{x}|^2), & x_1 > 0,\ x_2 > 0, \\ \quad 0 \quad , & \text{otherwise}, \end{cases}$$

as an illustration for the action of a two-dimensional operator built as the product $\mathcal{B}_{\mathrm{M}} = {}_1\mathcal{B}_{M_1}\,{}_2\mathcal{B}_{M_2}$ of one-dimensional operators ${}_i\mathcal{B}_{M_i}$ acting on the i-th argument of $\mathbf{x} = (x_1, x_2)$. These one-dimensional operators are based on the generating functions η_2 and η_6, which provide approximate order of convergence of $\mathcal{O}(h^2)$ and $\mathcal{O}(h^6)$, respectively. In similarity with the previous examples, we use $D = 3$ and step refinement ratio in all examples is $\mu^{-1} = 3$ in both the x_1 and x_2-directions. Again, the approximation results are plotted in logarithmic scales only in the interesting area near the vertex of the angle.

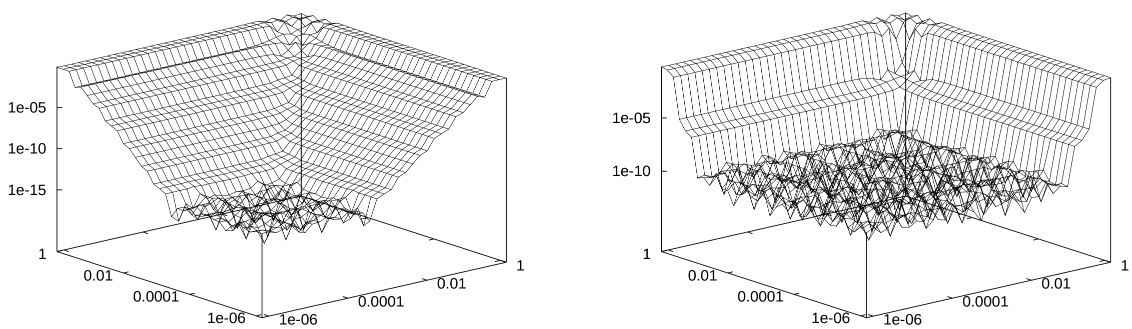

Figure 4: *Boundary layer error plots for the function* $\cos(100\,|\mathbf{x}|^2)$ *with support on the first quadrant of* $\mathbf{R}^2$ *using product of one-dimensional multi-resolution operators providing* **a)** *order* $\mathcal{O}(h^2)$ *of approximate convergence;* **b)** *order* $\mathcal{O}(h^6)$ *of approximate convergence.*

Precisely as in the one-dimensional examples, one can see clearly the gradual increase (Fig. 4a) of accuracy in the direction towards the boundary when the second-order formula is used. The plot in Fig. 4b shows the approximation results when a sixth-order formula is used. In this case the saturation level is reached already after two iterations.

7 Cubature of potentials in domains

In this section we derive some estimates for the cubature of integral operators, that often appear in problems of mathematical physics. As mentioned in the beginning, the cubature formula $P_h u$ for the integral operator

$$Pu(\mathbf{x}) = \int_\Omega k(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} \, .$$

is easily obtained from the boundary layer approximate approximations of the density u and defined as

$$P_h u(\mathbf{x}) := P\mathcal{B}_M u(\mathbf{x}) = \sum_{k=0}^{M} \sum_{h_k \mathbf{m} \in \mathcal{Q}_k} c_{k,\mathbf{m}} \int_{\mathbf{R}^n} k(\mathbf{x}-\mathbf{y})\eta\left(\frac{\mathbf{y}-h_k\mathbf{m}}{h_k\sqrt{D}}\right) d\mathbf{y} \, , \tag{38}$$

if η is chosen such that the integrals can be obtained analytically or by simple one-dimensional quadrature. For instance, the approximation by (38) of the harmonic potential $\mathcal{H}$ using the generating functions η_{2M} from (11) is obtained after calculating

$$\begin{aligned}
\mathcal{H}\eta_{2M}(\mathbf{x}) &= \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int_{\mathbf{R}^n} \frac{\eta_{2M}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\
&= \frac{1}{4|\mathbf{x}|^{n-2}\pi^{n/2}} \int_0^{|\mathbf{x}|^2} \tau^{n/2-2} e^{-\tau} \, d\tau + \pi^{-n/2} e^{-|\mathbf{x}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(n/2-1)}(|\mathbf{x}|^2)}{4(j+1)} \, .
\end{aligned}$$

Here $L_k^{(\alpha)}$ denote the generalized Laguerre polynomials (12). Some further examples for the action of different potentials of mathematical physics on the generating functions η_{2M} in any space dimension, including the elastic, hydrodynamic and diffraction potentials, can be found in [2], [3] and [9].

It is well known that many interesting operators are bounded mappings

$$P : L_p(\Omega) \to W_p^m(\Omega_1) \, , \tag{39}$$

with $\Omega, \ \Omega_1 \subset \mathbf{R}^n$; we write $P \in \mathcal{L}(L_p(\Omega), W_p^m(\Omega_1))$. Note that the case $m = 0$ corresponds to singular integral operators, whereas the volume potentials associated with partial differential equations satisfy relation (39) with $m > 0$. In any case the kernel function $k(\mathbf{x}-\mathbf{y})$ is singular at the diagonal $\mathbf{x} = \mathbf{y}$, so that the approximation of such multivariate integrals is quite complicated. If the operator P is such that (39) holds with $\Omega = \Omega_1 = \mathbf{R}^n$, Theorems 7 and 9 imply immediately:

Theorem 10 *Let $u \in W_p^N(\Omega)$ with $N > n/p$ and $P \in \mathcal{L}(L_p(\mathbf{R}^n), H_p^m(\mathbf{R}^n))$. For any $\varepsilon > 0$ there exists $D > 0$ such that*

$$||Pu - P_h u||_{H_p^m(\mathbf{R}^n)} \le c_1(Dh)^N ||\nabla_N u||_{L_p(\Omega)} + c_2(\mu^M h)^{1/p} ||u||_{L_\infty(\Omega)} + \varepsilon ||u||_{W_p^{N-1}(\Omega)} \, .$$

If additionally $P \in \mathcal{L}(H_p^{-m}(\mathbf{R}^n), L_p(\mathbf{R}^n))$ then

$$||Pu - P_h u||_{L_p(\mathbf{R}^n)} \le (c_1(Dh)^N + c_2(\mu^M h)^{1/p+r}) ||u||_{W_p^N(\Omega)} + \varepsilon \, h^m \, ||u||_{W_p^{N-1}(\Omega)} \, ,$$

where $0 < r < m/n$, $r \le (p-1)/p$.

However, very often the integral operator P fulfills (39) only for bounded domains $\Omega, \ \Omega_1 \subset \mathbf{R}^n$. Important examples are the harmonic or elastic potentials. In this case we are interested in the estimation of $Pu - P_h u$ on some bounded domain Ω_1. Since in general $\operatorname{supp} \mathcal{B}_M u = \mathbf{R}^n$ we have to consider also integrals of the form

$$\int_{\mathbf{R}^n \setminus \Omega} k(\mathbf{x}-\mathbf{y})\mathcal{B}_M u(\mathbf{y})d\mathbf{y} \, , \quad \mathbf{x} \in \Omega_1 \, .$$

To this end we choose a ball B_R with radius R around the origin such that $\Omega,\ \Omega_1 \subset B_R$ and suppose that the kernel satisfies the estimate

$$|\partial^{\boldsymbol{\alpha}} k(\mathbf{x}-\mathbf{y})| \le r_{\boldsymbol{\alpha}}(|\mathbf{y}|)\,, \quad \text{for } \ \mathbf{x} \in \Omega_1\,,\ \mathbf{y} \in \mathbf{R}^n \setminus B_R\,,$$

for some function $r_{\boldsymbol{\alpha}}(x)$ of at most polynomial growth and the multi-indexes $0 \le |\boldsymbol{\alpha}| \le m$.

Lemma 4 *For any $N > 0$ there exists constants $c_{N,\boldsymbol{\alpha},R}$ such that*

$$\Big\| \int_{\mathbf{R}^n\setminus B_R} \partial^{\boldsymbol{\alpha}} k(\cdot - \mathbf{y})\, \mathcal{B}_M u(\mathbf{y}) d\mathbf{y} \Big\|_{L_p(\Omega_1)} \le\ c_{j,\boldsymbol{\alpha},R}\, h^N (\text{meas } \Omega_1)^{1/p} \|u\|_{L_\infty(\Omega)}\,.$$

If $R \to \infty$ then $c_{N,\boldsymbol{\alpha},R} \to 0$.

PROOF. We estimate

$$\begin{aligned}
&\Big| \int\limits_{\mathbf{R}^n\setminus B_R} \partial^{\boldsymbol{\alpha}} k(\mathbf{y}-\mathbf{y}) \sum_{h_k\mathbf{m}\in\mathcal{Q}_k} c_{k,\mathbf{m}}\, \eta\left(\frac{\mathbf{y}-h_k\mathbf{m}}{h_k\sqrt{D}}\right) d\mathbf{y}\Big|^p \\
&\qquad \le c\|u\|^p_{L_\infty(\Omega)} \Big(\int\limits_{\mathbf{R}^n\setminus B_R} r_{\boldsymbol{\alpha}}(|\mathbf{y}|) \Big| \sum_{h_k\mathbf{m}\in\mathcal{Q}_k} \eta\left(\frac{\mathbf{y}-h_k\mathbf{m}}{h_k\sqrt{D}}\right) \Big| d\mathbf{y}\Big)^p \\
&\qquad \le c\|u\|^p_{L_\infty(\Omega)} \Big(\int\limits_{\mathbf{R}^n\setminus B_R} r_{\boldsymbol{\alpha}}(|\mathbf{y}|)\, g_{0,D}(\text{dist}(\frac{\mathbf{y}}{h_k}, \frac{\mathcal{Q}_k}{h_k})\, d\mathbf{y}\Big)^p .
\end{aligned}$$

Let $r_{\boldsymbol{\alpha}}(y) \le c_j |y|^j$ for $|y| \to \infty$. From the rapid decay of $g_{0,D}$ one obtains

$$g_{0,D}(\text{dist}(\frac{\mathbf{y}}{h_k}, \frac{\mathcal{Q}_k}{h_k})) = g_{0,D}(N_o + h_k^{-1}\,\text{dist}(\mathbf{y},\Omega)) \le c_N h_k^N\, \text{dist}(\mathbf{y},\Omega)^{-N}$$

for any N. Now it is clear that for $N > n+j$ the inequality

$$\int_{\mathbf{R}^n\setminus B_R} r_{\boldsymbol{\alpha}}(|\mathbf{y}|)\, g_{0,D}(\text{dist}(\frac{\mathbf{y}}{h_k}, \frac{\mathcal{Q}_k}{h_k}))\, d\mathbf{y} \le c h_k^N \int_{\mathbf{R}^n\setminus B_R} \frac{|\mathbf{y}|^j}{\text{dist}(\mathbf{y},\Omega)^N} d\mathbf{y}$$

proves the assertion. ∎

Now we are in a position to prove

Theorem 11 *Let $u \in W_p^N(\Omega)$ with $N > n/p$ and $P \in \mathcal{L}(L_p(\Omega), W_p^m(\Omega_1)$. Under the assumptions made above for any $\varepsilon > 0$ there exists $D > 0$ such that*

$$\|Pu - P_h u\|_{W_p^m(\Omega_1)} \le c_1 (Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2(\mu^M h)^{1/p} \|u\|_{L_\infty(\Omega)} + \varepsilon \|u\|_{W_p^{N-1}(\Omega)}\,.$$

If additionally $P \in \mathcal{L}((W^m_{p/(p-1)}(\Omega))', L_p(\Omega_1))$ then

$$\|Pu - P_h u\|_{L_p(\Omega_1)} \le (c_1(Dh)^N + c_2(\mu^M h)^{1/p+r}) \|u\|_{W_p^N(\Omega)} + \varepsilon\, h^m\, \|u\|_{W_p^{N-1}(\Omega)}\,,$$

where $0 < r < m/n$, $r \le (p-1)/p$.

PROOF. Fix the ball B_R and split

$$P_h u(\mathbf{x}) = P\mathcal{B}_M u(\mathbf{x}) = P\mathcal{X}_{B_R}\mathcal{B}_M u(\mathbf{x}) + P(1-\mathcal{X}_{B_R})\mathcal{B}_M u(\mathbf{x})\,.$$

The $W_p^m(\Omega_1)$-norm of the second term is bounded by $ch^N\|u\|_{L_\infty(\Omega)}$ due to Lemma 4, whereas the difference

$$\|Pu - P\mathcal{X}_{B_R}\mathcal{B}_M u\|_{W_p^m(\Omega_1)} \le c_R \|u - \mathcal{B}_M u\|_{L_p(B_R)}$$

can be estimated using by Theorem 7.

The same arguments apply also for the assertion concerning the $L_p(\Omega_1)$-norm of $Pu - P_h u$, if we use the inequality

$$\|Pu - P\mathcal{X}_{B_R}\mathcal{B}_M u\|_{L_p(\Omega_1)} \le c_R \|u - \mathcal{B}_M u\|_{(W^m_{p/(p-1)}(B_R))'} \le c\|u - \mathcal{B}_M u\|_{H_p^{-m}(\mathbf{R}^n)}$$

and Theorem 9. ■

Summarizing, for a large class of domain integral operators with singular kernels one can define cubature formulae retaining the order $O(h^N)$ plus some small saturation error, if the boundary layer approximate approximations of the density is used with appropriate parameters μ and M.

Let us consider two simple examples:

Example 1. Consider the logarithmic potential

$$\mathcal{H}_2 u(\mathbf{x}) = \frac{1}{2\pi}\int_\Omega u(\mathbf{y})\frac{1}{\log|\mathbf{x}-\mathbf{y}|}d\mathbf{y}$$

Note that the mapping

$$\mathcal{H}_2 : L_p(\Omega) \mapsto W_p^2(\Omega), \quad 1 < p < \infty,$$

is bounded, if Ω is a bounded domain. Thus, in this case Theorem 11 yields the estimate

$$\|\mathcal{H}_2 u - \mathcal{H}_{2,h} u\|_{W_p^2(\Omega)} \le c_1(Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2(\mu^M h)^{1/p}\|u\|_{L_\infty(\Omega)} + \varepsilon\|u\|_{W_p^{N-1}(\Omega)} .$$

Consequently, if the boundary layer approximate approximations are such that μ^M is of the same order of magnitude as h^{Np-1}, we get the approximation order $O(h^N)$ modulo saturation error. If we measure the error in a weaker norm than W_p^2, the small saturation error tends to zero together with h. For example, if $u \in W_2^N(\Omega)$ with $N > 1$ we obtain

$$\|\mathcal{L}_2 u - \mathcal{L}_{2,h} u\|_{L_2(\Omega)} \le (c_1(Dh)^N + c_2\mu^M h)\|u\|_{W_2^N(\Omega)} + \varepsilon\, h^2\, \|u\|_{W_2^{N-1}(\Omega)} ,$$

such that already the choice $\mu^M \asymp h^{N-1}$ leads to $O(h^N)$ order plus a very small error term converging to zero with the rate $O(h^2)$. Note that Sobolev's imbedding theorem can be used to prove the convergence of the cubature $\mathcal{L}_{2,h}$ with respect to the uniform norm.

Example 2. The Poisson integral

$$\mathcal{P}_n\varphi(\mathbf{x},t) = \frac{1}{(2\sqrt{\pi t})^n}\int_{\mathbf{R}^n}\varphi(\mathbf{y})\exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|}{4t}\Big)dy , \quad \mathbf{x}\in\mathbf{R}^n ,\ t>0 ,$$

gives a partial solution of the homogeneous heat equation with the initial condition $u(\mathbf{x},0) = \varphi(\mathbf{x})$. If the basis function η is the Gaussian or some related function then obviously the integrals

$$\frac{1}{(2\sqrt{\pi t})^n}\int_{\mathbf{R}^n}\eta\left(\frac{\mathbf{y}-h_k\mathbf{m}}{h_k\sqrt{D}}\right)\exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|}{4t}\Big)d\mathbf{y}$$

have simple analytic expressions. Since for fixed $t > 0$ the kernel function is smooth the Poisson integral generates a bounded mapping from Sobolev or Bessel potential spaces of arbitrary negative order into usual function spaces. Therefore from Theorem 11 it follows that

$$\|\mathcal{P}\varphi(\cdot,t) - \mathcal{P}_h\varphi(\cdot,t)\|_{L_2(\mathbf{R}^n)} \le (c_1(Dh)^N + c_2\mu^M h)\|\varphi\|_{W_2^N(\Omega)} ,$$

with constants depending on $t > 0$ but not on φ and h. Hence, $\mathcal{P}_h$ represents a semi-analytic cubature of order $O(h^N)$ without saturation errors.

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